Course Name: Essentials of Topology
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Welcome to Lecture 4 on Essentials of Topology. Continuing with the theory of sets and functions, in this lecture, we will study the fundamental notions of associated set functions, which play a key role in the study of continuous functions between topological spaces. Actually, for a given function f, we will find the image and inverse image of sets under the function. Let us see this concept through the diagram. The first one is for a function $f : X \to Y$, which sends each element of the domain to some element of co-domain. The question is, by using this function, is it possible to compute the image of a subset of X and an image of a subset of Y? The answer is yes.

What we can do for a given subset A of X, we can compute its image by using the function f. Also, for a given subset of Y, we can compute its image again using the same function f. So what exactly happens for any function $f : X \to Y$? There exist two functions: one sends a subset of X to a subset of Y, and another sends a subset of Y to a subset of X. Formally, for a given function $f : X \to Y$, there exist two functions, one function from the power set of X to the power set of Y, which sends each subset A of X to $f(A)$, called image of A under f, where $f(A)$ is defined as:

$$f(A) = \{f(x) : x \in A\}.$$

Also, there exists another function from the power set of Y to the power set of X, which sends each $B \subseteq Y$ to $f^{-1}(B)$, and this $f^{-1}(B)$ is called the inverse image of B under f, where $f^{-1}(B)$ is defined as:

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

It is to be noted here that whenever we use this notation $f^{-1}(B)$, this is only a notation that differs from the concept of the inverse of a function. Note that in case of the existence of an inverse of a function, the function should be bijective, but here we are starting with an arbitrary function. From these two definitions, it is clear that $f(A)$, i.e., the image of A under function f, is always a subset of Y, while the inverse image of B under f is always a subset of X. These are simple observations.

The question is, what about the image of an empty set and the inverse image of an empty set? Both will be empty sets; just think about it. It's simple. We have already seen that for all $A \subseteq X$, $f(A)$ is always a subset of Y, so it is natural to say that $f(X)$ will always be a subset of Y, but the question is, what about equality? The answer is simple. $f(X)$ is equal to Y if and only if f is surjective, and this is trivial.

Finally, $f^{-1}(Y)$ will always be equal to X. The question is, why? Just recall the definition that $f^{-1}(Y)$ will be a collection of elements of X such that $f(x) \in Y$ because this holds for every element of X. Moving ahead, let us take some examples. The first is regarding finding out the image of a subset of X by using a given function f. So, the function f is defined as:

$$f(a) = p, f(b) = q, f(c) = q.$$

If we want to find out what is the image of subsets of X under the function. So, the image of the empty set will be an empty set; this is simple. Image of a singleton set $\{a\}$ will be a singleton set $\{p\}$, because $f(a) = p$. Image of $\{b\}$ will be a singleton set $\{q\}$ because $f(b) = q$. Image of a singleton set $\{c\}$ will be a singleton set $\{q\}$ because $f(c) = q$.

If we want to find out what about the image of set $\{a, b\}$, so as $f(a)$ is p and $f(b)$ is q, the image of set $\{a, b\}$ will be $\{p, q\}$, and that is Y (Read B as Y in the video) itself. If we want to find out the image of set $\{b, c\}$, the value of $f(b)$ is q, and $f(c)$ is also q, therefore the image of this subset $\{b, c\}$ of X will be set $\{q\}$. As $f(a)$ is equal to p and $f(c)$ is equal to q, therefore f image of $\{a, c\}$ will be set Y, and if we want to find out the image of X (read A as X, in video) under f, as $f(a)$ is p, $f(b)$ is q, and $f(c)$ is also q, therefore $f(X)$ is equal to Y (read B as Y). It is clear that this is a well-defined function from the power set of X to the power set of Y. Moving ahead, for the same function f defined from X to Y, let us compute $f^{-1}(B)$, where B is a subset of Y.

This is clear that $f^{-1}(\emptyset) = \emptyset$. The question is, what about the inverse image of a singleton set $\{p\}$? As $f(a) = p$, therefore the inverse image of a singleton set $\{p\}$ is a singleton set $\{a\}$. Since $f(b) = q$ and $f(c)$ is also q, by using the concept of the inverse image, the image of a singleton set $\{q\}$ will be two points set $\{b, c\}$, while the inverse image of Y will be X itself. Again, we can see that this is a well-defined function. Let us take some more examples; let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = x^2$, then what will be the image of a singleton set $\{3\}$.

This is clear from the definition of function, as $f(x) = x^2$, so $f(\{3\}) = \{9\}$. If we want to find out $f^{-1}(\{-9\})$, as none of the elements of the domain can be mapped to negative real numbers, therefore the inverse image will be an empty set. Moving ahead, let us compute the image of set $\{-3, 3\}$, because $f(-3) = 9$ and $f(3) = 9$, therefore the image of this set will be a singleton set $\{9\}$. If we want to find out an inverse image of singleton set $\{9\}$, since -3 and 3 both can be mapped to 9, therefore this inverse image will be $\{-3, 3\}$. Moving ahead, if we are trying to find out the image of a closed interval $[-3, 3]$, it is clear that this image will be nothing but a closed interval $[0, 9]$, while the inverse image of $[0, 9]$ will be $[-3, 3]$.

Let us take one more example: the image of a closed interval $[-3, 1]$. Is it $[0, 9]$? Need to think and the answer is yes, and if we are looking for $f^{-1}([-1, 9])$, that will again be $[-3, 3]$. So, what we have seen is that for a given function f from the set of real numbers to the set of real numbers, we have found an image of some subsets of the set of real numbers and an inverse image of some of the subsets of real numbers. Moving ahead, let us study some of the properties of these functions. The first question is if f is a function from X to Y, $A \subseteq X$, and $B \subseteq Y$. Then what about the existence of $f^{-1}(f(A))$ as well as $f(f^{-1}(B))$?

Note that both concepts exist because we have already seen that if we are taking a function from the power set of X to the power set of Y and another from the power set of Y to the power set of X, then for every subset A of X, this function send it to $f(A)$ and for every subset $f(A)$ of Y this will go to $f^{-1}(f(A))$. So we can talk about the first one. Similarly, if we are taking functions from the power set of Y to the power set of X and the power set of X to the power set of Y, i.e., if we are taking a subset B of Y, then the function f^{-1} will send it to $f^{-1}(B)$ and after that, we can compute its image, that will be $f(f^{-1}(B))$. This means that we can talk about both the concepts.

Let us try to relate these concepts with the original set, that is, with A as well as with B. Begin with the first one. For any function f from X to Y and $A \subseteq X$, A is always a subset of $f^{-1}(f(A))$. This is simple to justify as for any $x \in A$, $f(x) \in f(A)$, or $x \in f^{-1}(f(A))$, showing that A is subset of $f^{-1}(f(A))$. But the question comes that under what condition does equality hold?

Answer is, $A = f^{-1}(f(A))$ if and only if f is injective. Let us see. The first one is, let us assume that f be injective. Then, our motive is to prove that $A = f^{-1}(f(A))$, and this is equivalent to proving two things.

First one is $A \subseteq f^{-1}(f(A))$, and $f^{-1}(f(A)) \subseteq A$. It is to be noted here that this holds for every subset A of X, which we have already seen. So, we only want to justify that the $f^{-1}(f(A))$ is a subset of A, for which let any $x \in f^{-1}(f(A))$. Then it is clear that $f(x) \in f(A)$, and if $f(x) \in f(A)$, what we can do that this $f(x)$ can be written as $f(x')$, for $x' \in A$. But note that, our function is injective, and because of the injectivity of the function, $x = x'$, and note that $x' \in A$.

So, we can conclude that $x \in A$, and therefore, $f^{-1}(f(A))$ is a subset of A, and we can conclude that $A = f^{-1}(f(A))$. Let us see the converse part of this result. Meaning is that if for all subsets A of X, given that $A = f^{-1}(f(A))$, then we have to show that f is injective. For which, let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Then, to show the injectivity of the function, we have to justify that $x_1 = x_2$.

For which, let A be a singleton set $\{x_1\}$. Then what about the image of A under f? There will be nothing but singleton set $\{f(x_1)\}$. But as $f(x_1) = f(x_2)$, we can conclude that $f(x_2) \in f(A)$, or $x_2 \in f^{-1}(f(A))$. But as per our assumption $f^{-1}(f(A))$ is equal to A, or this is simply $\{x_1\}$, or that x_2 belongs to set $\{x_1\}$, that is $x_1 = x_2$, and therefore f is an injective function. So, we have justified that $A \subseteq f^{-1}(f(A))$ if and only if f is injective.

Moving ahead, we can also justify that for a given function $f : X \to Y$ and a subset B of Y, $f(f^{-1}(B)) \subseteq B$. Answer is, if we are taking any $y \in f(f^{-1}(B))$, then we can say that $y = f(x)$, where $x \in f^{-1}(B)$. Meaning is that, $f(x) \in B$, but note that y equals $f(x)$, so we can conclude that $y \in B$. So, we have shown that $f(f^{-1}(B)) \subseteq B$. It is to be noted here that, in general, $f(f^{-1}(B))$

may not equal to B.

For example, let us take a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Also, take B as $[-9, 9]$. Then what about $f^{-1}(B)$? $f^{-1}(B)$ is nothing, but this is a closed interval $[-3, 3]$, and if we are computing $f(f^{-1}(B))$, this is nothing but $[0, 9]$, which is a proper subset of B. So, this example justified that, in general, $f^{-1}(B)$ may not equal B. The question comes under which condition will equality hold?

The answer is here. $f(f^{-1}(B)) = B$ if and only if f is surjective. Let us justify it. So, first, assume that f is surjective. Our motive is to prove that $f(f^{-1}(B)) = B$, which can be proved in two parts. The first one is, $f(f^{-1}(B)) \subseteq B$, and the second one is $B \subseteq f(f^{-1}(B))$.

It is clear from the first one, which we have already shown, that $f(f^{-1}(B)) \subseteq B$, for all subsets B of Y. So, it only remains to justify that $B \subseteq f(f^{-1}(B))$. Let us prove it. So, take any $y \in B$. Note that this is already given to us that f is a surjective function, and as B is a subset of Y, there exists an element x in the domain such that $f(x) = y$.

It means that $f(x) \in B$, and if $f(x) \in B$, we can conclude that $x \in f^{-1}(B)$, or $f(x) \in f(f^{-1}(B))$, or $y \in f(f^{-1}(B))$, and therefore, B is a subset of $f(f^{-1}(B))$. So, we have shown our requirement, and therefore, $f(f^{-1}(B)) = B$. Let us prove the converse part of the result. Let $f(f^{-1}(B)) = B$, for all $B \subseteq Y$. Then we have to justify that f is surjective, for which let us take any $y \in Y$ and B as a singleton set $\{y\}$.

Then from the assumption, $f(f^{-1}(\{y\})) = \{y\}$, or we can say that $y \in f(f^{-1}(\{y\}))$, or y is equal to $f(x)$, for $x \in f^{-1}(\{y\})$, or we can say that there exist $x \in f^{-1}(\{y\})$, and that is nothing but a subset of X such that $y = f(x)$, and therefore f is a surjective function.

These are the references.

That's all from today's lecture. Thank you.