Course Name: Essentials of Topology
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Week: 07
Lecture: 04

Welcome to Lecture 39 on Essentials of Topology.

In this lecture, we will also continue the study of the concept of connectedness. Specifically, we will study the connectedness of the set of reals and its subsets. We have seen that $\mathbb{R}$ with the lower limit topology is disconnected. We have also seen that if we are taking $\mathbb{R}$ with Euclidean topology, $\mathbb{Q} \subseteq \mathbb{R}$, that is, the set of rationals, this is disconnected. Till now, we have not discussed the connectedness of $\mathbb{R}$ with Euclidean topology. So, first, our motive is to discuss the connectedness of $(\mathbb{R}, \mathcal{T}_e)$, and after that, we will see some of the connected subsets of $\mathbb{R}$. In order to prove that $(\mathbb{R}, \mathcal{T}_e)$ is connected, we require two fundamental notions that are well-known in analysis. The first one is that every subset of $\mathbb{R}$, which is bounded above, has a least upper bound, and this least upper bound is also known as supremum. This concept is known as the least upper-bound property. The second one is, if we are taking two real numbers, x and y, with $x < y$, then there exists a real number z such that $x < z < y$. With these two fundamental concepts, let us show that $\mathbb{R}$, along with the Euclidean topology, is connected. We will prove it by contradiction.

If possible, let $(\mathbb{R}, \mathcal{T}_e)$ be disconnected. Then, there exists $A, B \subset \mathbb{R}$ such that these are nonempty, A and B are open, $A \cap B = \emptyset$, and $A \cup B = \mathbb{R}$. Now, as A and B are nonempty, we can take a real number a in A and another real number b in B. Note that $a, b \in \mathbb{R}$, so either $a < b$, or $a = b$, or $a > b$. But it is to be noted that A and B are disjoint. Thus, $a \neq b$. Therefore, either $a < b$, or $a > b$. Here, we are taking $a < b$. One can discuss for $a > b$, similarly. Now, let us take $A_0 = [a, b] \cap A$ and $B_0 = [a, b] \cap B$. These A_0 and B_0 have some interesting features given as: A_0 and B_0 are nonempty, because $a \in [a, b] \cap A$ and $b \in [a, b] \cap B$. Also, A_0 and B_0 are open subsets of $[a, b]$. Further, $A_0 \cap B_0 = [a, b] \cap (A \cap B) = \emptyset$. Finally, $A_0 \cup B_0 = ([a, b] \cap A) \cup ([a, b] \cap B) = [a, b] \cap (A \cup B) = [a, b]$, as $A \cup B = \mathbb{R}$. Thus, what we have seen is that if the real line with Euclidean topology is disconnected, we can also create a separation of the closed interval $[a, b]$, and that separation is given by the pair (A_0, B_0).

Moving ahead, as we have taken $a \in A$, it is to be noted that A is $\mathcal{T}_e$-open. Therefore, for small $\epsilon > 0$, $[a, a+\epsilon) \subseteq [a, b] \cap A = A_0$. Similarly, as $b \in B$, and B is also $\mathcal{T}_e$-open, $(b-\epsilon, b] \subseteq B \cap [a, b] = B_0$. We have seen that $A_0 \cup B_0 = [a, b]$. Meaning is, $A_0 \subseteq [a, b]$. If $A_0 \subseteq [a, b]$, we can say that A_0 is bounded above. If this is bounded above, by using the least upper bound property, there exists the supremum of the set A_0. Let us take the supremum of A_0 as c. Also, as $A_0 \subseteq [a, b]$, we can say that this c will always lie between a and b. Now, what can we observe? We can observe that $a + \epsilon \leq c \leq b - \epsilon$. The question is, why? The answer is, what c is? c is the supremum of A_0, and A_0 is a superset of $[a, a+\epsilon)$. What is the supremum of $[a, a+\epsilon)$? That is $a+\epsilon$. Therefore, this $a+\epsilon \leq c$. Now, the question is: why $c \leq b - \epsilon$? Note that if we are taking any $x \in A_0$, whether $x > b - \epsilon$. The answer is no. Why? Because if $x > b - \epsilon$, then $x \in (b - \epsilon, b] \subseteq B_0$; that is, $x \in B_0$. But what we have seen is that A_0 and B_0 are disjoint, therefore this is a contradiction. It means that none of the elements of this set, that is, A_0 can be greater than $b - \epsilon$. That is $b - \epsilon$ is an upper bound of A_0. That's why $c \leq b - \epsilon$. Finally, what do we have with us? Note that $c \in [a, b] = A_0 \cup B_0$. We can show that c is neither an element of A_0 nor an element of B_0. If we can justify these two, then from here, we reach a contradiction, and after that, we can conclude that $(\mathbb{R}, \mathcal{T}_e)$ is connected. So, we are going to justify that $c \notin A_0$ and $c \notin B_0$.

If $c \in A_0$, which is precisely $[a, b] \cap A$. Then $c \in A$. Note that A is an open subset of $\mathbb{R}$. Therefore, as we have discussed above, we can find $\delta > 0$ such that $[c, c+\delta) \subseteq A \cap [a, b] = A_0$. Now, $c + \delta/2 \in [c, c+\delta) \subseteq A_0$. It means that $c < c + \delta/2 \in A_0$. But c is the supremum of A_0. Thus, we reach a contradiction. Therefore, $c \notin A_0$. Again, if $c \in B_0$. Then as $B_0 = [a, b] \cap B$, $c \in B$. What is this B? B is an open subset of $\mathbb{R}$. So, similar to the justification that we have used above, we can conclude that there exists some $\delta' > 0$ such that $(c - \delta', c] \subseteq B \cap [a, b] = B_0$. Now, as $c - \delta'/2 \in (c - \delta', c] \subseteq B_0$. Therefore, $c - \frac{\delta'}{2} \in B_0$. As we have justified that $b - \epsilon$ is an upper bound of A_0, in the same line, we can say that $c - \frac{\delta'}{2}$ is an upper bound of A_0. But note that c is the least upper bound, and what we have with us, this $c - \frac{\delta'}{2} < c$. Again, we reach a contradiction. Thus, $c \notin B_0$. Therefore, $(\mathbb{R}, \mathcal{T}_e)$ is connected.

Having the knowledge that the real line with Euclidean topology is connected, we can talk about the connectedness of $\mathbb{R}^n$ with standard topology or Euclidean topology on it. It follows from the fact that we have already studied that if we have connected topological spaces $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$, ..., $(X_n, \mathcal{T}_n)$, then the product space $(X_1 \times X_2 \times ... \times X_n, \mathcal{T})$, is also connected. As we have already justified, the real line with Euclidean topology is connected. Therefore, $\mathbb{R}^n$ with the standard topology is also connected.

Moving ahead, using the fact that the real line with Euclidean topology is connected, we can justify that all the intervals are connected. Even if we are beginning with $\{a\}$, where $a \in \mathbb{R}$, note that $\{a\}$ can be visualized as $[a, a]$, and we know that singleton sets are always connected. Therefore, $[a, a]$ is connected. Now, if we are taking the open interval (a, b), where $a < b$, and $a, b \in \mathbb{R}$. Then, as we have already seen that (a, b) is homeomorphic to $\mathbb{R}$, (a, b) is a connected set as the connectedness is a topological property and $\mathbb{R}$ with the Euclidean topology is connected. Moving ahead, we know that if a set is connected, its closure is also connected. So, as this open interval (a, b) is connected, the closure of (a, b), i.e., $[a, b]$ is also connected. Also, we know that this $(a, b) \subset [a, b) \subset [a, b]$, and $(a, b) \subset (a, b] \subset [a, b]$. Therefore, we can also conclude that the semi-open intervals are also connected. Moving ahead, we have also shown that (a, ∞) is homeomorphic to $\mathbb{R}$. Again, with the same justification as above, we can say that (a, ∞) a connected set. Now, if (a, ∞) is connected, what about its closure? Its closure is nothing but $[a, \infty)$. Therefore, $[a, \infty)$ is also connected. Finally, as (a, ∞) is homeomorphic to $(-\infty, a)$, meaning that the intervals of the form $(-\infty, a)$ are also connected. If $(-\infty, a)$ is connected, what about its closure? If we are taking the closure of $(-\infty, a)$, that is nothing but $(-\infty, a]$, this is also connected. Thus, all the intervals are connected.

Moving ahead, we can justify that connected subsets of $\mathbb{R}$ are only intervals. That is, if we are taking $A \subseteq \mathbb{R}$, which is not an interval, then A is disconnected. Let us assume that A is connected, and if possible, let A be not an interval. Also, take two real numbers a and b in A such that $a < b$. Because A is not an interval, there exists $x \in \mathbb{R}$ such that $a < x < b$ and $x \notin A$. Now, note that $(-\infty, x)$ and (x, ∞) are $\mathcal{T}_e$-open. So, let us take $A_0 = A \cap (-\infty, x)$, and $B_0 = A \cap (x, \infty)$. Then A_0 and B_0 are nonempty, because $a \in A_0$ and $b \in B_0$. Further, A_0 and B_0 are open in A, as we can visualize A as a subspace of $\mathbb{R}$ with the Euclidean topology. If we are looking for the intersection of A_0 and B_0, it is clear that A_0 and B_0 are disjoint. Finally, $A_0 \cup B_0 = (A \cap (-\infty, x)) \cup (A \cap (x, \infty)) = A$. Thus, we have created a separation, which is a pair (A_0, B_0) of A. But note that we have already assumed that A is connected. So, this is a contradiction. We reached this contradiction because we assumed that A was not an interval. Hence, A is an interval.

These are the references.

That's all from this lecture. Thank you.