Course Name: Essentials of Topology
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Welcome to Lecture 38 on Essentials of Topology.

We will continue the study of the concept of connectedness in this lecture, too. Specifically, we will focus on the study of the concept of a component, which is a maximal connected set in a topological space.

The idea behind this concept is, for example, if we have a space $(X, \mathcal{T})$, The question is: can we make a partition of this space? The answer is yes, we can do it. Whenever we make partitions, note that they should be in the form of connected subsets. We are looking for the largest connected subsets, not only connected subsets. Whenever we are thinking about the partition of any set, we look towards the concept of equivalence relations. Thus, by using the concept of equivalence relation, we can also talk about the partition of a topological space in the form of connected subsets. Let us discuss through an example, take $X = \{a, b, c\}$ with discrete topology. We know that only singleton sets are connected subsets in discrete spaces. So, $\{a\}$, $\{b\}$, and $\{c\}$ are connected. Even $X = \{a\} \cup \{b\} \cup \{c\}$. Note that the singleton sets are not only connected; they are the largest connected sets containing the elements.

Now, let us have a detailed idea of how to define components by using the concept of equivalence relations. Begin with a topological space $(X, \mathcal{T})$. Define a relation R on X as: For $x, y \in X$, xRy if $x, y \in C$, for some connected subset C. We can see that R is an equivalence relation. If we are looking for the reflexivity of the relation, note that xRx, as we know that singleton set $\{x\}$ is always connected such that $x \in \{x\}$. R is symmetric, because if xRy, then $x, y \in C$, for some connected subset C, or $y, x \in C$, i.e., yRx. Finally, for the transitivity of this relation, let us take xRy and yRz. Note that if xRy, then there exists a connected set C_1 such that $x, y \in C_1$. Similarly, if yRz, we can find a connected subset C_2 such that $y, z \in C_2$. Now, our motive is to construct a connected set containing x and z to justify the transitivity of the relation. Note that if we are looking at the structure of C_1 and C_2, their intersection is nonempty, i.e., $C_1 \cap C_2 \neq \emptyset$, therefore $C_1 \cup C_2$ is connected. Also, the interesting feature of $C_1 \cup C_2$ is that it contains x and z, both. Thus, xRz, or that R is an equivalence relation. Also, if R is an equivalence relation, we can talk about the equivalence class of an element x of X, and the equivalence class is given by $[x] = \{y \in X : xRy\} = \{y \in X : x, y \in C, C$ is a connected set in $X\}$. Thus, we can talk about the partition of the space, that is, $\bigcup_{x \in X} [x]$. Finally, if $(X, \mathcal{T})$ is a topological space and R is an equivalence relation on X such that xRy if $x, y \in C$, for some connected subset C of space $(X, \mathcal{T})$. Then, the equivalence class is called a component in X. We will use the notation $C(x)$ to denote the equivalence class of x.

Moving ahead, let us see some characterizations of a component. The first one is: if we have a topological space $(X, \mathcal{T})$, and $x \in X$, then the component $C(x)$ is a union of all connected sets containing x, i.e., $C(x) = \cup\{C : C$ is a connected set containing $x\}$. In order to justify it, let $y \in C(x)$. Then y and x are related; that is, there exists a connected set C such that $x, y \in C$. From here, it is clear that this C contains x, and C contains y. So, we can conclude that $y \in \cup\{C : C$ is a connected set containing $x\}$. Now, let $y \in \cup\{C : C$ is a connected set containing $x\}$. Then there exists a connected set; let us take this as C'. This C' has the property that C' contains x, and this also contains y. If C' is containing both x and y, we can say that xRy, or we can say that $y \in C(x)$. Thus, $C(x) = \cup\{C : C$ is a connected set containing $x\}$. From here, it is clear that every connected set in the union is containing x. So, their intersection is nonempty, and therefore $\cup\{C : C$ is a connected set containing $x\}$ is a connected set. Thus, the component $C(x)$ is a connected subset of X. Also, we can conclude from here that $C(x)$ is a maximal connected set containing x.

Moving ahead, we can justify that if we are taking a connected set A in the topological space $(X, \mathcal{T})$, then $A \subseteq C(x)$, for some $x \in X$. It is clear from the definition of component. Still, if we want to see it, let $x \in A$. Then $x \in C(x)$, as $C(x)$ is an equivalence class containing x. Now, if $y \in A$, then as $x \in A$, and A is a connected set, obviously x and y are related, that is $y \in [x] = C(x)$. Thus, $A \subseteq C(x)$, for some $x \in X$. Moving ahead, the component $C(x)$ is a closed set. Why? The answer is, we know that every set is a subset of its closure. So, $C(x) \subseteq \overline{C(x)}$. Note that we have shown that $C(x)$ is connected, therefore $\overline{C(x)}$ is also connected. But $C(x)$ is not only connected; this is a maximal connected set containing x. Therefore, $C(x) = \overline{C(x)}$, and hence $C(x)$ is a closed set.

Moving ahead, let us see some of the examples on components. The first example is: let us take indiscrete topological space $(X, \mathcal{T})$. As indiscrete spaces are connected. So, in this case, there will only be one component, and that component is X itself, i.e., for all $x \in X$, $C(x) = X$. Let us take another example of the real line with co-finite topology. This space is also connected. Therefore, for all $r \in \mathbb{R}$, $C(r) = \mathbb{R}$. If we are taking the example of discrete space $(X, \mathcal{T})$. As, we know that the singleton sets are only connected sets in discrete spaces; So, for all $x \in X$, $C(x) = \{x\}$. Moving ahead, let us take one more example of the real line with the lower limit topology. We know that this space is disconnected. Even, what we can justify is that any subset of $A \subseteq \mathbb{R}$ containing more than one element will always be disconnected. For example, if we are taking $A = \{x, y\}$, we can make a separation of this set. As $x, y \in \mathbb{R}$, we can find $z \in \mathbb{R}$ such that $x < z < y$. Note that we can make a separation of $\mathbb{R}$, by using two intervals $(-\infty, z)$ and $[z, \infty)$; which are the members of lower limit topology. If we are taking $A \cap (-\infty, z) = A_0$ and $A \cap [z, \infty) = B_0$. Then $A_0 \cup B_0 = A \cap ((-\infty, z) \cup [z, \infty)) = A \cap \mathbb{R} = A$. Also, $A_0 \cap B_0 = A \cap ((-\infty, z) \cap [z, \infty)) = A \cap \emptyset = \emptyset$. Also, A_0 contains x, and B_0 contains y. Therefore, A_0 and B_0, both are nonempty, these A_0 and B_0 are open subsets of A. So, we have created a separation of A, therefore A is disconnected. Thus, for all $r \in \mathbb{R}$, $C(r) = \{r\}$.

Even, if we are taking another example, let us take the set of real numbers with Euclidean topology. Let us take the set of rational numbers $\mathbb{Q}$. Note that we can justify that $\mathbb{Q}$ is disconnected. Even, for all $q \in \mathbb{Q}$, if we are looking for the component, that will be again a singleton set $\{q\}$. We have seen a number of examples here. Some of the interesting examples, like the case of lower limit topology on $\mathbb{R}$, we have seen that the components are singleton sets, and if the components are singleton sets, such topological spaces have a special name. We say that a topological space $(X, \mathcal{T})$ is totally disconnected, if for all $x \in X$, $C(x) = \{x\}$. The first such example is the real line with lower limit topology. The second example, $(\mathbb{R}, \mathcal{T})$, where $\mathcal{T}$ is discrete topology. Also, if we are taking the set of real numbers with Euclidean topology and $\mathbb{Q} \subset \mathbb{R}$, then $\mathbb{Q}$ is also totally disconnected.

Moving ahead, let us see the behavior of components under homeomorphism. We can justify that if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a homeomorphism, then the image of a component in X is a component in Y. That is, if we are taking a homeomorphism $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$, $x \in X$, and a component $C(x) = C$, the $f(C)$ is a component in Y. In order to justify it, begin with, as C is a component, it is connected. As f is a homeomorphism, f is continuous, too, therefore $f(C)$ is a connected subset of Y. But our motive is to justify that $f(C)$ is a component. As we have already studied that every connected set is always a subset of a component, so let us take D as a component in Y such that $f(C) \subseteq D$. Now, we can show that $f(C) = D$. In order to justify it, let us use the continuity of f^{-1}; this is continuous because f is a homeomorphism. If f^{-1} is continuous and D is a component in Y, we can say that $f^{-1}(D)$ is a connected subset of X. It is also to be noted that $f(C) \subseteq D$, or that $C \subseteq f^{-1}(D)$. Note that C is a component and $f^{-1}(D)$ is a connected subset of X. But as the component is a maximal connected set, therefore, by the maximality of C, $C = f^{-1}(D)$, or $f(C) = D$. Therefore, $f(C)$ is a component.

These are the references.

That's all from this lecture. Thank you.