

**Course Name: Essentials of Topology**  
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Welcome to Lecture 37 on Essentials of Topology.

In this lecture too, we will continue the study of the concept of connectedness. Specifically, we will see the connectedness of the union of two connected sets or an arbitrary family of connected sets; we will also study the connectedness of a continuous image of a connected topological space. Further, we will continue to study the connectedness of product of connected topological spaces.

Begin with the concept of the union of connected subsets of a topological space. Let us see an example. If we are having a set  $X = \{a, b, c\}$ , and the topology  $\mathcal{T}$  on  $X$  is discrete. Now, let us take two subsets  $A_1 = \{a\}$  and  $A_2 = \{b\}$  of  $X$ . Note that the singleton sets are always connected in a topological space. Therefore,  $A_1$  and  $A_2$  are connected. But if we are taking their union, that is,  $A_1 \cup A_2 = \{a, b\}$ . In discrete space, this is disconnected. From this example, it is clear that the union of two connected subsets may not be connected. So, the question is: Can we put some conditions in place such that the union of connected sets becomes connected? The answer is yes, given in the form of following theorem.

The theorem states as: Let  $(X, \mathcal{T})$  be a topological space and  $\{E_i : i \in I\}$  be a collection of connected subsets of  $X$  such that  $\bigcap_{i \in I} E_i \neq \emptyset$ . Then  $\bigcup_{i \in I} E_i$  is connected. In order to prove it, let us take  $E = \bigcup_{i \in I} E_i$ , and assume that  $E$  is disconnected. If  $E$  is disconnected, what will happen? We can find two subsets  $A$  and  $B$  of  $E$  such that  $A$  and  $B$  are nonempty,  $A$  and  $B$  are open subsets of  $E$ , they are disjoint, and their union is  $E$ . Now, as  $\bigcap_{i \in I} E_i \neq \emptyset$ , let  $x \in \bigcap_{i \in I} E_i$ . Then, it is clear that  $x \in E_i$ , for all  $i \in I$ . If  $x \in E_i$ ,  $x \in E = A \cup B$ . But  $A$  and  $B$  are disjoint. So, it is clear that either  $x \in A$  or  $x \in B$ . Note that  $x$  cannot be an element of both  $A$  as well as  $B$ . Now, as  $E = \bigcup_{i \in I} E_i$ , and every  $E_i \subseteq E$ , as this  $E$  has a partition. Therefore, we can conclude that  $E_i \subseteq A$  or  $E_i \subseteq B$ ,

and this is for all  $i \in I$ . Now, if we are taking  $x \in A$ , then we can conclude that  $E_i \subseteq A, \forall i \in I$ . If every  $E_i \subseteq A$ , we can conclude that  $E = \bigcup_{i \in I} E_i \subseteq A$ , or  $E \subseteq A$ . But what have we seen? We have taken a separation of  $E$  in terms of  $A$  as well as  $B$ , where  $B$  is nonempty. So, it contradicts that the pair  $(A, B)$  is a separation of  $E$ . Similarly, if we are considering the second case, that is,  $x \in B$ , we can conclude that  $E_i \subseteq B, \forall i \in I$ . With the same justification as above, we can conclude that  $E \subseteq B$ , and it contradicts the fact that the pair  $(A, B)$  is a separation of  $E$ . Hence, in any case, if we assume that  $E$  is disconnected, we reach a contradiction. Therefore,  $E = \bigcup_{i \in I} E_i$  is connected.

Moving ahead, herein we will discuss the connectedness of continuous images of a connected topological space. This theorem states that if  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$  is a continuous function and  $(X, \mathcal{T})$  is connected. Then,  $f(X)$  is a connected subset of  $Y$ . Again, we will prove it by contradiction. Let, if possible,  $f(X)$  be disconnected. Then there exist two subsets,  $A$  and  $B$  of  $f(X)$  such that  $A$  and  $B$  are nonempty,  $A$  and  $B$  are open in  $f(X)$ , they are disjoint, and  $f(X)$  is the union of  $A$  as well as  $B$ . As the function  $f$  is continuous,  $f^{-1}(A)$ , and  $f^{-1}(B)$  are  $\mathcal{T}$ -open. Also, as  $A$  and  $B$  are disjoint, we can conclude that  $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ . Finally,  $X = f^{-1}(A \cup B)$ ; that is,  $X = f^{-1}(A) \cup f^{-1}(B)$ . Further, as  $A$  and  $B$  are nonempty,  $f^{-1}(A) \neq \emptyset$ , and  $f^{-1}(B) \neq \emptyset$ . From here, what can we conclude? We can conclude that the pair  $(f^{-1}(A), f^{-1}(B))$ , is a separation of the topological space  $(X, \mathcal{T})$ . It means that  $(X, \mathcal{T})$  is disconnected, which is a contradiction. Therefore,  $f(X)$  is a connected subset of  $Y$ . Moving ahead, the above theorem can also be written as the continuous image of a connected topological space is connected. In case if we are taking  $f(X) = Y$ , meaning to say that  $f$  is a surjective function, then if  $f$  is a continuous surjective function from a topological space  $(X, \mathcal{T})$  to  $(Y, \mathcal{T}')$ , where  $(X, \mathcal{T})$  is connected, then  $(Y, \mathcal{T}')$  is also connected. Thus, using it, we can conclude that connectedness is a topological property; that is, if we are taking a homeomorphism from a topological space  $(X, \mathcal{T})$  to  $(Y, \mathcal{T}')$ , and we are taking  $(X, \mathcal{T})$  as a connected topological space, then it is clear that  $(Y, \mathcal{T}')$  is also connected.

Using the above concept, we can justify that the quotient space of a connected topological space is connected. The justification is simple because we are familiar with the concept of a quotient map  $q$ , which is given as  $q : (X, \mathcal{T}) \rightarrow (A, \mathcal{T}_q)$ , where  $\mathcal{T}_q$  is quotient topology on  $A$ . We have already seen that  $q$  is a con-

tinuous and surjective function. Also, if we are taking  $(X, \mathcal{T})$  as a connected topological space. Then, from the previous conclusion, we can deduce that  $(A, \mathcal{T}_q)$  is also connected.

Moving ahead, let us discuss the connectedness of product space. Begin with two topological spaces  $(X_1, \mathcal{T}_1)$ , and  $(X_2, \mathcal{T}_2)$ , which are connected. We can show that the product space, that is,  $X_1 \times X_2$ , along with the product topology, is connected. In order to prove this theorem, let us begin with a simple notion. If we are taking an element  $x_2 \in X_2$ , then  $X_1 \times \{x_2\} \subseteq X_1 \times X_2$  with the relative topology with respect to the product topology on  $X_1 \times X_2$  is a subspace. It is to be noted here that we can define a function  $f : X_1 \rightarrow X_1 \times \{x_2\}$  such that  $f(x_1) = (x_1, x_2), x_1 \in X_1$ . It is clear that this function is a bijective function. Also, it can be seen that this is continuous and open, too. Just think about it. If  $f$  is a bijective, continuous, and open function, we can conclude that  $X_1 \times \{x_2\}$  is homeomorphic to the space  $(X_1, \mathcal{T}_1)$ , or let us write only  $X_1$ . Similarly, we can construct another space  $\{x_1\} \times X_2$ , where  $x_1 \in X_1$ . Again, we can define a function  $g : X_2 \rightarrow \{x_1\} \times X_2, x_2 \in X_2$  such that  $g(x_2) = (x_1, x_2)$ . Note that  $g$  is a bijective, continuous, and open function. So, we can conclude the space  $\{x_1\} \times X_2$  is homeomorphic to  $X_2$ . This is the key step for proving that the product space is connected. How will it play the role? We already know that  $(X_1, \mathcal{T}_1)$ , and  $(X_2, \mathcal{T}_2)$ , are connected topological spaces. As connectedness is a topological property,  $X_1 \times \{x_2\}$  homeomorphic to  $(X_1, \mathcal{T}_1)$ , and  $\{x_1\} \times X_2$  is homeomorphic to  $(X_2, \mathcal{T}_2)$ . Therefore, we can conclude that  $X_1 \times \{x_2\}$ , and  $\{x_1\} \times X_2$ , both are connected. Note that  $(X_1 \times \{x_2\}) \cap (\{x_1\} \times X_2) \neq \emptyset$  as  $(x_1, x_2)$  is a member of this intersection. Now, if  $(X_1 \times \{x_2\}) \cap (\{x_1\} \times X_2) \neq \emptyset$ ,  $(X_1 \times \{x_2\}) \cup (\{x_1\} \times X_2)$ , is connected. We have already seen that the union of connected sets is connected, provided their intersection is nonempty. Now, by using the connectedness of this union, for fix  $a \in X_2$ , and  $x \in X_1$ , let  $C_x = (X_1 \times \{a\}) \cup (\{x\} \times X_2)$ . It is clear that for all  $x \in X_1$ , the intersection of  $C_x$  is nonempty. Therefore,  $\cup\{C_x : x \in X_1\}$  is connected as this is the union of connected sets with nonempty intersections. But the interesting fact is,  $\cup\{C_x : x \in X_1\} = X_1 \times X_2$ . Therefore,  $X_1 \times X_2$  with product topology is connected.

Further, if we have a finite number of connected topological spaces  $(X_1, \mathcal{T}_1)$ ,  $(X_2, \mathcal{T}_2)$ , ...,  $(X_n, \mathcal{T}_n)$ . Then, we can generalize our result, and we can conclude that the product space, that is,  $X_1 \times X_2 \times \dots \times X_n$ , is also connected. We

can use this result to discuss the connectedness of  $\mathbb{R}^n$ . But before that, we require the concept of the connectedness of the set of real numbers, which we will discuss in the next lecture.

These are the references.

That's all from this lecture. Thank you.