Course Name: Essentials of Topology
Professor Name: S.P. Tiwari
Department Name: Mathematics & Computing
Institute Name: Indian Institute of Technology(ISM), Dhanbad
Week: 06
Lecture: 05

Welcome to Lecture 35 on Essentials of Topology.

In this lecture, we will study the concept of connectedness of topological spaces. We will continue to study this concept in a series of lectures. Connectedness plays an important role in the study of topological spaces, which we will see. It also plays the key role behind the well-known Intermediate Value Theorem. From the name itself, it seems that we are talking about a space whose partition cannot be made, that is, which cannot be split into two parts. Meaning is to say that if we are having a topological space $(X, \mathcal{T})$, let us take the set X like this one, and we cannot make a partition of this set, that is, it cannot be written in the form of something like $X = A \cup B$, $A \cap B$ is an empty set. Obviously, if we are taking two parts, so both parts should be non-empty. But the question is, where does the topology play the role? This is only the set-theoretic approach, even if we are taking the set of real numbers. Let us take this as the element 0. Now, let us take a set $A = (-\infty, 0)$, and another set $B = [0, \infty)$. Note that $\mathbb{R}$ can be expressed as $A \cup B$, and $A \cap B$ is an empty set. Also, neither A nor B is empty. The question is, is it a separation of $\mathbb{R}$? The answer is no because still, we are not using the power of topology. What is going on is that 0 is the limit point of A and $0 \in B$. Note that whenever we are talking about the separation, we have to make a restriction in addition to this one that the set should be disjoint, plus their limit points cannot move from one part to another.

Begin with the formal definition of connectedness. Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is disconnected if there exist $A, B \subset X$ such that

- $A \neq \emptyset, B \neq \emptyset$;
- A and B are open;
- $A \cap B = \emptyset$; and
- $X = A \cup B$.

Whenever there exists such A and B, we say that this pair makes a separation of X. Now, having the idea of this disconnectedness, what is the meaning of connectedness? It's a simple that a topological space $(X, \mathcal{T})$ is connected if $(X, \mathcal{T})$ is not disconnected.

We were discussing that the limit points of A cannot go in B, and the limit points of B cannot be a member of A, but when we are defining the concept of separation for a disconnected topological space, we are taking that A and B are open. The question is, from here, how can we deduce whether the limit points of A cannot be members of B or limit points of B cannot be members of A? It is simple to justify. Because we are taking $A \cap B$ as an empty set, and $X = A \cup B$. From here, we can conclude that $A = B^c$. Note that because B is open, B^c is closed, or that A is closed, and if A is closed, we can say that the $cl(A) = A$. Therefore, as $A \cap B$ is an empty set, we can say that $cl(A) \cap B$ is an empty set. Also, we know that $cl(A) = A \cup A'$. Now, $A' \cap B \subseteq cl(A) \cap B = \emptyset$. Similarly, we can talk about B, which can be expressed as A^c, and we can conclude that $A \cap B' = \emptyset$; that is, neither the limit points of A can be members of B nor limit points of B can be members of A.

Let us see some of the examples. Our first example is if we are taking a set $X = \{a, b, c\}$. Let us take two topologies on X; our first topology is $\mathcal{T}_1 = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$. Let us also take another topology, $\mathcal{T}_2 = \{\emptyset, X, \{a, b\}, \{c\}\}$. From here, we conclude that this $(X, \mathcal{T}_1)$ is connected because we cannot find two non-empty disjoint subsets of X whose union is equal to X, and they are open, too. But note that in the case of the second one, that is $(X, \mathcal{T}_2)$, this is disconnected. The question is, why? The answer is that X can be written as $\{a, b\} \cup \{c\}$. Note that $\{a, b\}$ and $\{c\}$ are open. Also, these sets are disjoint; that is, their intersection is an empty set, and obviously, these are non-empty. Thus, we have taken an example of a connected space, while on the same set if we are changing the topology, the space becomes disconnected.

Let us take another example. We know the Sierpinski topology. Let us take $X = \{a, b\}$ and $\mathcal{T} = \{\emptyset, X, \{a\}\}$. Note that this topological space $(X, \mathcal{T})$ is connected because we cannot find two disjoint non-empty open subsets whose union is equal to X. Let us take one more example. Let us take a non-empty set X, and the topology $\mathcal{T}$ on X is indiscrete. Note that $\mathcal{T} = \{\emptyset, X\}$, and from the definition of topology, it is clear that X cannot be expressed in the form of a union of two disjoint non-empty open subsets. Therefore, indiscrete topological spaces are connected.

Let us take some of the examples of disconnected spaces. The first one is, why not let us take the discrete topology $\mathcal{T}$ on X, where $|X| \geq 2$. This will always be disconnected. Why? The answer is simple. Let us take a proper subset A of X, which is non-empty. Then obviously, A^c will also be non-empty, and this is a subset of X. Because topology is discrete, A and A^c are open. Also, we know that $X = A \cup A^c$, and $A \cap A^c = \emptyset$. Therefore, this topological space is disconnected. Moving ahead, let us take another example. Let us take the real line with the lower limit topology. In this topology, let us take two parts of the real line. If we are taking $A = (-\infty, 0)$ and $B = [0, \infty)$. Note that $\mathbb{R} = A \cup B$, $A \cap B$ is an empty set. Also, neither A nor B is empty, and both A and B are members of the lower limit topology. Note that this B is not a member of Euclidean topology and that we are not discussing. Even the real line with Euclidean topology is connected, which we will see later. So, because A and B are open and all required conditions are satisfied, the real line with the lower limit topology is disconnected.

Moving ahead, let us see a characterization of disconnectedness. After the characterization of disconnectedness, we can conclude something about connectedness, as the concept of connectedness is based on disconnectedness. This is given in the form of a theorem, and the statement of the theorem is: a topological space $(X, \mathcal{T})$ is disconnected if and only if there exists a non-empty proper subset of X, which is both open and closed. In order to prove it, let us assume that $(X, \mathcal{T})$ is disconnected. If we are saying that $(X, \mathcal{T})$ is disconnected. What we have already seen is that there exists some subsets A and B of X such that these are non-empty, these are open, these A and B are disjoint, and X can be expressed as $A \cup B$. Now, from here, we can conclude that $A = B^c$. As this B is an open set, we can say that A is closed. Also, as A is open, so we can say that A is both open as well as closed. Also, note that A is non-empty and B is non-empty. Therefore, A is a proper subset of X, which is non-empty too. Thus, we have shown that there exists a non-empty proper subset of X, which is both open and closed.

Moving to the other part, let us see the converse of this theorem. For which, let us assume that there exists a non-empty proper subset A of X, which is both open and closed. Then, we have to justify that the topological space $(X, \mathcal{T})$ is disconnected. Note that if A is closed, we can say that A^c is open. Also, A is a proper subset of X. Therefore, A^c is non-empty, and obviously, this is a subset of X. Thus, what have we got? We got two subsets, A and A^c, these are subsets of X such that A^c is closed. Note that A is non-empty, and A^c is also non-empty. Also, A and A^c are open. From set theory, we know that $A \cap A^c = \emptyset$ and $A \cup A^c = X$. Thus, we have created a separation of X, and therefore, this topological space $(X, \mathcal{T})$ is disconnected.

Moving to the next concept, let us see the characterization of connectedness. We say that a topological space is connected if and only X and $\emptyset$ are only open and closed sets in $(X, \mathcal{T})$. Justification is simple because whenever we are talking about the space $(X, \mathcal{T})$ is connected, it means this is not disconnected, and if this is not disconnected, it means that no non-empty proper subset of X is closed and open. Therefore, X and $\emptyset$ are only open and closed. If we are talking about the non-empty subsets, only X will be the non-empty subset, which will be open and closed, and if we are justifying in the reverse direction, we can conclude that $(X, \mathcal{T})$ is connected. For example, let us take a set $X = \{a, b, c\}$, take a topology $\mathcal{T}_1 = \{\emptyset, X, \{a\}\}$ on it. If we are looking for closed sets here, the closed sets will be X, $\emptyset$, and $\{b, c\}$. Note that except X and $\emptyset$, no other sets are closed as well as open. So, what we can say that this $(X, \mathcal{T}_1)$ is connected. But if we are making a change in topology, let us take this topology $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$. If we are computing the closed sets here, then the closed sets are X, $\emptyset$, $\{b, c\}$, and $\{a\}$. Note that this singleton set $\{a\}$ is open as well as closed. Therefore, this topological space $(X, \mathcal{T}_2)$ is disconnected.

Moving ahead, let us take some more examples. Why not let us talk about this real line with co-finite topology? Note that this topological space is connected. Let us justify by contradiction. If possible, there is a proper subset A of $\mathbb{R}$, which is non-empty, and this A is closed and open both. Note that if A is closed, this A is a finite set, and if this is open, what about A^c? A^c is closed. We know that in co-finite space, closed sets are finite sets; therefore, A^c is finite. Now, as $\mathbb{R} = A \cup A^c$ with the condition that A and A^c both are finite. It means this $\mathbb{R}$ itself is finite, which is not possible, that is a contradiction. So, our assumption is wrong, and therefore, this topological space, that is co-finite topological space is connected.

Moving ahead, let us see the characterization of connectedness in terms of a continuous function. This is one of the interesting characterizations of connected topological spaces. The statement of this theorem is given as: a topological space $(X, \mathcal{T})$ is connected if and only if every continuous function from $(X, \mathcal{T})$ to a discrete topological space is constant. In order to prove this theorem, let us first assume that the given topological space $(X, \mathcal{T})$ is connected. Now, our motive is to justify that every continuous function from $(X, \mathcal{T})$ to a discrete topological space is constant. So, let us assume a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$. Note that this $\mathcal{T}'$ is discrete. If this is a discrete topology on Y, for all $y \in Y$, the singleton set $\{y\}$ is both open and closed. Now, let us take $y_0 \in Y$ such that $y_0 \in Im(f)$. If $y_0 \in Im(f)$, then $f^{-1}(\{y_0\})$ cannot be empty. Also, by continuity of the function f, $f^{-1}(\{y_0\})$ is open and closed. Now, if $f^{-1}(\{y_0\})$ is a proper subset of X and this is non-empty, then this is a proper subset that is both closed and open, and in that case, the topological space becomes disconnected. But as the given topological space is connected, this cannot be a proper subset of X, and therefore, this $f^{-1}(\{y_0\}) = X$, or we can say that $f(X) = \{y_0\}$, for all $x \in X$, or this f is a constant function. That is the proof of this part.

Let us see the converse part of this theorem. For this, we assume that every continuous function from the topological space $(X, \mathcal{T})$ to a discrete topological space is constant. Our motive is to prove that the topological space $(X, \mathcal{T})$ is connected. We are going to justify it by contradiction. So, if possible, let us assume that $(X, \mathcal{T})$ is disconnected, and if this space is disconnected, we can talk about the separation of it; that is, there exist subsets A and B of X such that these are non-empty. Also, A and B are open. This $A \cap B$ is the empty set, and $X = A \cup B$. Now, let us take this $Y = \{a, b\}$. Also, take the discrete topology on it, that is, $\mathcal{T}' = \{\emptyset, Y, \{a\}, \{b\}\}$. Now, we are defining a function $f : X \to Y$, such that $f(A) = \{a\}$ and $f(B) = \{b\}$. What about this function? We can say that this f is continuous. Why? The answer is: $f^{-1}(\emptyset) = \emptyset, f^{-1}(Y) = X, f^{-1}(\{a\}) = A$, and $f^{-1}(\{b\}) = B$. If this is so, we have defined a function that is not a constant function, but that is continuous. Therefore, we reached a contradiction, and hence, the topological space $(X, \mathcal{T})$ is connected.

Moving ahead, till now we have discussed about the connectedness or disconnectedness of a topological space. A question arises that if the topological space is connected and we are taking a subset A of this X, whether A is also connected. The answer is, not necessarily. It may happen, or we can provide some examples when subsets in a connected topological space may not be connected. The question is how? Before going to the formal definition of the connectedness of a subset, let us see an example that if topological space is connected, there exists some subsets of it that may not be connected, that is, which is disconnected. Let us take co-finite topological space $(X, \mathcal{T})$. This will serve our purpose, but for that, we require one more characterization that is if this topological space is co-finite and if we are taking any subset Y of X, the question is what about the subspace $(Y, \mathcal{T}_Y)$, whether this space is also co-finite. The answer is yes, and we can justify it. After this justification, we will discuss the connectedness. We know that $\mathcal{T}_Y = \{Y \cap G : G \in \mathcal{T}\}$. We can show that $H \in \mathcal{T}_Y$ if and only if H^c is finite, and if we can justify it, then we can conclude that $(Y, \mathcal{T}_Y)$ is co-finite. So, let us take $H \in \mathcal{T}_Y$. Then, $H = Y \cap G$, where $G \in \mathcal{T}$. Now, $H^c = Y - (Y \cap G) = Y \cap (Y \cap G)^c = Y \cap (Y^c \cup G^c) = (Y \cap Y^c) \cup (Y \cap G^c) = Y \cap G^c$, which is finite.

Now, let us see the converse part. If we are taking $H \subseteq Y$ such that H^c is finite, so why not let us take $H = Y - F$, where F is a finite subset of Y. Note that if F is a finite subset of Y, this is also a finite subset of X, and if this is a finite subset of X, $X - F \in \mathcal{T}$. Now, $Y \cap (X - F) = Y \cap (X \cap F^c) = Y \cap F^c = Y - F = H$. So, we can conclude that every H can be expressed as $Y \cap G$, where $G = Y - F \in \mathcal{T}$. Therefore, H is a member of $\mathcal{T}_Y$. So, we have justified that the subspace of a co-finite topological space is also co-finite. Now, let us make a change here if we are taking this $(X, \mathcal{T})$, where $\mathcal{T}$ is co-finite topology and X is an infinite set. This is always a connected topological space. Now, let us take $Y \subseteq X$, but Y is finite. We know that if we are discussing co-finite topology on a finite set, what will happen? This $(Y, \mathcal{T}_Y)$ is a discrete topological space and we have already discussed that if $|Y| \geq 2$, a discrete topological space is disconnected. So, what we have seen is that a subset of a connected topological space is disconnected. It is to be noted here that whenever we are talking about the connectedness of this subset Y of X, we are discussing in the framework of relative topology. So, from here, we can discuss or we can define the connectedness of a subset, and actually, that is defined in terms of subspace.

Let us see the formal definition of the connectedness of a subset in a topological space. Let $(X, \mathcal{T})$ be a topological space, and E be a subset of X. We say that E is connected if the subspace $(E, \mathcal{T}_E)$ is connected. Note that if we are having the concept of connectedness, we can also talk about the disconnectedness of E, that is, E is disconnected, if this topological space, that is subspace is disconnected. Let us take some of the examples. Begin with a topological space $(X, \mathcal{T})$. Note that the singleton set $\{x\} \subseteq X$, for all $x \in X$, is connected. This is simple because we cannot make a separation of a singleton set.

Let us take some more examples. Let us take co-finite topological space $(X, \mathcal{T})$, where X is an infinite set. Let us take a subset Y of X, where Y is also an infinite set. If we are taking the relative topology on Y, we have seen that this is co-finite. As an infinite set with co-finite topology, it is always connected. Therefore, this $(Y, \mathcal{T}_Y)$ is also connected. It means that if we are taking a co-finite topological space, every infinite subset of X is connected.

Moving ahead, let us take some of the examples which are disconnected. Begin with the real line with Euclidean topology. Let us take a set X, which is the union of two intervals $(-1, 0)$, and $(0, 1)$. Note that in relative topology, if we are taking $(-1, 0)$ as A and $(0, 1)$ is B, both A and B are open. That is, X can be written as a union of two sets. Also, $A \cap B$ is an empty set. A is non-empty, B is non-empty, and therefore this X is disconnected. Let us take another example. Again, $\mathbb{R}$ is equipped with Euclidean topology. Let us take a real number p, and we are taking a subset $\mathbb{R} - \{p\}$ of $\mathbb{R}$. How will this subset look like? If we are taking A as $(-\infty, p)$ and B as (p, ∞). Note that this $\mathbb{R} - \{p\}$ can be written as $A \cup B$. Further, $A \cap B$ is an empty set. Obviously, A is non-empty, and B is non-empty. Also, A and B are both open in relative topology. Therefore, we have created a separation of this set, and hence this set is disconnected.

These are the references.

That's all from this lecture. Thank you.