

**Course Name: Essentials of Topology**  
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Welcome to Lecture 33 on Essentials of Topology.

In this lecture too, we will study the concept of homeomorphisms. Begin with the concept of a homeomorphism. In the previous lecture, we saw that whenever we are talking about a homeomorphism  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$ , we have seen that this  $f$  was a bijective and bi-continuous function, in the sense that this  $f$  is continuous and  $f^{-1}$  is also continuous. When we talked about the continuity of  $f^{-1} : Y \rightarrow X$ , we have seen that for  $H \in \mathcal{T}$ ,  $(f^{-1})^{-1}(H) \in \mathcal{T}'$ . But at the same time, because of the bijectiveness, we can say that  $f(H) \in \mathcal{T}'$ . So, what precisely is going on that instead of computing the inverse of the inverse of  $f$ , we are finding out the direct image of this  $H$ , that is  $f(H)$ , and if  $H$  is  $\mathcal{T}$ -open implies  $f(H)$  is  $\mathcal{T}'$ -open, that is equivalent to say that  $f^{-1}$  is continuous. The functions sending  $\mathcal{T}$ -open sets to  $\mathcal{T}'$ -open sets are also known as open functions. Thus, in this lecture, we will study the characterization of homeomorphisms in terms of open functions. When we talk about open functions, the question is, what about closed functions? So, we will also discuss about it. We will discuss the possible relationship between open and closed functions and their characterizations in terms of interior as well as closure.

Begin with, let  $(X, \mathcal{T})$  and  $(Y, \mathcal{T}')$  be topological spaces. Then a function  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$  is

- open, if image of each  $\mathcal{T}$ -open set is a  $\mathcal{T}'$ -open set; and
- closed, if image of each  $\mathcal{T}$ -closed set is a  $\mathcal{T}'$ -closed set.

Let us take some of the examples of open and closed functions. Begin with the first example, let us take a function, this is an arbitrary function  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$ , where this  $\mathcal{T}'$  is a discrete topology. Now, if we are taking any  $G \in \mathcal{T}$ ,  $f(G) \subseteq Y$ . Note that the topology on  $Y$  is discrete. Therefore  $f(G) \in \mathcal{T}'$ . Thus, the function  $f$  is open. Also, let us take a closed subset of  $X$ ; that is, if we are taking this  $F$  as a  $\mathcal{T}$ -closed subset of  $X$ , then what about  $f(F)$ , which

will always be a subset of  $Y$ , and if this is a subset of  $Y$ , note that the topology on  $Y$  is discrete. Therefore,  $f(F)$  is  $\mathcal{T}'$ -closed, or we can say that the function  $f$  is closed. So, this function is both open as well as closed.

Let us take another example. For example, we are taking this open interval  $(2, 3)$ , this is a subset of  $\mathbb{R}$ . Let us take  $\mathbb{R}$  equipped with Euclidean topology  $\mathcal{T}_e$ . So, we are assuming  $(2, 3)$  as a subspace of this  $\mathbb{R}$  with Euclidean topology. Now, let us take the inclusion function  $i : (2, 3) \rightarrow \mathbb{R}$ . Note that in the case of relative topology, the open intervals of the form  $(a, b)$  will be open with respect to the topology on this open interval  $(2, 3)$ . Because we are taking this as an inclusion function, we can say that the image of such open interval  $(a, b)$  under this  $i$ , that will be  $(a, b)$ , and this is a member of the Euclidean topology. Therefore, the function  $i$  is open. It is to be noted here that this  $(2, 3)$  is also closed with respect to the relative topology. But note that if we are finding out its image, that is, the image of this  $(2, 3)$  or closed set with respect to relative topology, this will always be this  $(2, 3)$ , which is not  $\mathcal{T}_e$ -closed. Therefore, the function  $i$  is not closed. So, this inclusion function is open but not closed.

Let us take one more example. Let us take a function  $f : (\mathbb{R}, \mathcal{T}_e) \rightarrow (\mathbb{R}, \mathcal{T}_e)$  such that  $f(x) = c$  (a constant), for all  $x \in \mathbb{R}$ . Note that this function is not open, but it is closed. Why is this not open? The answer is, for example, if we are taking  $(2, 3) \in \mathcal{T}_e$ ,  $f((2, 3)) = \{c\}$ , which is not  $\mathcal{T}_e$ -open. Therefore, this function is not open. But if we are taking any closed subset with respect to Euclidean topology, that is,  $F$  is  $\mathcal{T}_e$ -closed,  $f(F) = \{c\}$ , which is always  $\mathcal{T}_e$ -closed. Therefore, this function is closed. So, we have discussed a number of examples, and we have seen that the functions may be open and closed. Also, open functions may not be closed, and a closed function may not be open.

Moving ahead, let us discuss some examples regarding open as well as continuous functions. We can construct some functions which are open but not continuous, and at the same time, one can also construct some functions, which are continuous but not open. Now, let us take the first example. This is a function  $f : (\mathbb{R}, \mathcal{T}_e) \rightarrow (\mathbb{R}, \mathcal{T}_l)$  such that  $f(x) = x$ , for all  $x \in \mathbb{R}$ . Then if we are taking any  $G \in \mathcal{T}_e$ ,  $f(G) = G \in \mathcal{T}_l$ , as we know that Euclidean topology is coarser than lower limit topology. Therefore, this  $f$  is open, but note that  $f$  is not continuous because if we are taking  $[2, 3) \in \mathcal{T}_l$ , then  $f^{-1}([2, 3))$  is not a member of  $\mathcal{T}_e$ . So, this is an example of a function which is open but not

continuous.

Let us construct another function from this example. Let us take a function  $f : (\mathbb{R}, \mathcal{T}_l) \rightarrow (\mathbb{R}, \mathcal{T}_e)$  such that  $f(x) = x$ , for all  $x \in \mathbb{R}$ . Now, if we are taking any  $G \in \mathcal{T}_e$ , then  $f^{-1}(G) = G \in \mathcal{T}_l$ , with the same justification as Euclidean topology is coarser than the lower limit topology. Therefore this  $f$  is a continuous function. But if we are taking  $[2, 3) \in \mathcal{T}_l$ , then  $f([2, 3))$  is not a member of  $\mathcal{T}_e$ . Therefore,  $f$  is not open. So, what have we seen? We have seen that there may be functions that are open but not continuous, and there exist functions that are continuous but not open. Even, we can construct some functions which can be open and continuous.

Moving ahead, finally we can characterize homeomorphisms in terms of open and closed functions. The characterization is given by this theorem: Let  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$  be a bijective continuous function. Then  $f$  is a homeomorphism iff  $f$  is open iff  $f$  is closed. The justification is simple because we already have a bijective continuous function with us. In order to justify it, if we are beginning with a homeomorphism  $f$ , it means that this is bijective continuous, which is already given, plus one additional thing, which is that  $f^{-1}$  is continuous. Note that  $f^{-1}$  is continuous, this is equivalent to say that  $(f^{-1})^{-1}(G) \in \mathcal{T}$ , when  $G \in \mathcal{T}'$ , and this is equivalent to say that  $f(G) \in \mathcal{T}'$ , and this is equivalent to say that the function  $f$  is open. So,  $f$  is a homeomorphism if and only if  $f$  is open. Now, if we are assuming that  $f$  is open, we can justify that  $f$  is closed and vice-versa. Now, let us assume that  $f$  is open and let us take a subset  $F$  of  $X$ , which is  $\mathcal{T}$ -closed. Now, if this is  $\mathcal{T}$ -closed, we can say that  $X - F$  is  $\mathcal{T}$ -open. Because this function is open, therefore,  $f(X - F)$  is  $\mathcal{T}'$ -open. As the function is bijective, we can say that  $Y - f(F)$  is  $\mathcal{T}'$ -open, or this  $f(F)$  is  $\mathcal{T}'$ -closed, or simply the function  $f$  is closed. Similarly, if we are taking the function  $f$  as closed, we can justify that  $f$  is open.

Moving ahead, let us see a characterization of a closed function. If we are having two topological spaces  $(X, \mathcal{T})$  and  $(Y, \mathcal{T}')$ , then the function  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$  is closed if and only if  $\overline{f(A)} \subseteq f(\bar{A})$ , for all  $A \subseteq X$ . The justification is simple. Let us assume that this function is closed. Now, what we have to do that we have to prove that  $\overline{f(A)} \subseteq f(\bar{A})$ . In order to justify it, as we know,  $A \subseteq \bar{A}$ , and this is true for all  $A \subseteq X$ . So,  $f(A) \subseteq f(\bar{A})$ , or  $\overline{f(A)} \subseteq \overline{f(\bar{A})}$ . But note that because function  $f$  is closed and  $\bar{A} \subseteq X$  is  $\mathcal{T}$ -closed,  $f(\bar{A})$  is

$\mathcal{T}'$ -closed because of closedness of  $f$ . Therefore,  $\overline{f(\bar{A})} = f(\bar{A})$ . Thus, we can conclude here that  $\overline{f(A)} \subseteq f(\bar{A})$ . That's the proof of it.

In order to prove the converse part, let us assume  $\overline{f(A)} \subseteq f(\bar{A})$ , for all  $A \subseteq X$  and try to justify that  $f$  is a closed function. In order to prove that  $f$  is a closed function, let us take  $F$  is  $\mathcal{T}$ -closed. We are assuming it, and we have to show that  $f(F)$  is  $\mathcal{T}'$ -closed. In order to justify that it is  $\mathcal{T}'$ -closed, it is enough to show that  $\overline{f(F)} = f(F)$ , and this is equivalent to these two statements that justify that  $f(F) \subseteq \overline{f(F)}$ , and also justify that  $\overline{f(F)} \subseteq f(F)$ . First always holds as  $A$  is always a subset of  $\bar{A}$ . In order to justify second, let us use the assumption that we have with us. If we are replacing this  $A$  by  $F$ , then we can get  $\overline{f(F)} \subseteq f(\bar{F})$ . But note that we have already assumed that  $F$  is  $\mathcal{T}$ -closed. Because of this,  $\bar{F} = F$ . Therefore, we can write that  $\overline{f(F)} \subseteq f(F)$ . That's the justification, and therefore,  $\overline{f(F)} = f(F)$ . Hence, this function  $f$  is closed.

Moving ahead, let us see the characterization of a homeomorphism. Just recall, we have already seen the characterization of a continuous function in terms of closure. Also, we have seen the characterization of closed function in terms of closure. It is natural that by combining these two, we can characterize homeomorphism in terms of closure. If we are having two topological spaces  $(X, \mathcal{T})$  and  $(Y, \mathcal{T}')$ . Then a bijective function  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$  is a homeomorphism if and only if  $\overline{f(A)} = f(\bar{A})$ , for all  $A \subseteq X$ . Note that we have seen that the function  $f$  is continuous if and only if  $f(\bar{A}) \subseteq \overline{f(A)}$ , for all  $A \subseteq X$ . Also, we have seen that  $f$  is a closed function if and only if  $\overline{f(A)} \subseteq f(\bar{A})$ , for all  $A \subseteq X$ . Note that we have already assumed that the function is bijective, so homeomorphism meaning is that this is continuous and closed and if this is continuous and closed, this is a combination of  $f(\bar{A}) \subseteq \overline{f(A)}$  and  $\overline{f(A)} \subseteq f(\bar{A})$ . Thus,  $\overline{f(A)} = f(\bar{A})$ , and that is the proof of this theorem.

Similar to the concept of characterization of closed functions, we can characterize open functions. We have seen the characterization of closed function in terms of closure. Obviously, as open sets are defined in terms of interior, so we can justify or we can characterize an open function in terms of interior. Let us see the characterization. The characterization is given by this theorem and the statement is: Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{T}')$  be topological spaces. Then  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$  is open iff  $f(A^\circ) \subseteq (f(A))^\circ$ , for all  $A \subseteq X$ . In order to justify it, let us first assume that  $f$  be open. Now, let us take  $A \subseteq X$ , we know

that  $A^\circ \subseteq A$ . Therefore,  $f(A^\circ) \subseteq f(A)$ , or we can say that  $(f(A^\circ))^\circ \subseteq (f(A))^\circ$ . But note that the function  $f$  is open, and if  $f$  is open, we know that  $A^\circ \subseteq X$ , this is  $\mathcal{T}$ -open, and if this is  $\mathcal{T}$ -open, by openness of  $f$ ,  $f(A^\circ)$  is  $\mathcal{T}'$ -open, and if this is  $\mathcal{T}'$ -open what we can conclude from here, that is  $(f(A^\circ))^\circ = f(A^\circ)$ , that is  $f(A^\circ) \subseteq (f(A))^\circ$ , this is the proof of the first part of this theorem.

Moving ahead, now let us assume that  $f(A^\circ) \subseteq (f(A))^\circ$ , for all  $A \subseteq X$  and try to justify that  $f$  is open. Now, if we want to prove that this function  $f$  is open, let us take any  $A \subseteq X$ , where  $A \in \mathcal{T}$ . Our motive is to justify that  $f(A) \in \mathcal{T}'$ , or just prove that  $(f(A))^\circ = f(A)$ . This is equivalent to show that  $(f(A))^\circ \subseteq f(A)$ , and  $f(A) \subseteq (f(A))^\circ$ . Note that this first always holds because we know that for every subset  $P$  of  $Y$ ,  $P^\circ \subseteq P$ . In order to prove the second one, let us use our assumption. The assumption is given as  $f(A^\circ) \subseteq (f(A))^\circ$ . But note that what have we taken? We have taken  $A$  as a  $\mathcal{T}$ -open set; that is,  $A^\circ = A$ . Therefore, this  $f(A) \subseteq (f(A))^\circ$ , which is the justification of this statement. Therefore,  $(f(A))^\circ = f(A)$ , or  $f(A)$  is  $\mathcal{T}'$ -open. Hence, the function  $f$  is open.

These are the references.

That's all from this lecture. Thank you.