Course Name: Essentials of Topology
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Institute Name: Indian Institute of Technology(ISM), Dhanbad
Week: 06
Lecture: 02

Welcome to Lecture 32 on Essentials of Topology.

In this lecture, we will study the concept of homeomorphisms. Recall the concept of isomorphism in algebra. An isomorphism between groups or rings, which we have already studied, is a bijective map that preserves algebraic structures present therein. The question arises, what about such functions in the case of topological spaces, which preserve topological structures? The answer is given in terms of homeomorphism. Note that when we are talking about preserving the topological structures, we know that the members of topology are known as open sets. The question comes: how can we define a bijective function from one topological space to another topological space that preserves open sets? The answer is, this is something like if we are having a bijective function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$. Let us assume that this function is continuous too; then we know that if we are taking $G \in \mathcal{T}'$, $f^{-1}(G) \in \mathcal{T}$. But the question is, if we are taking an open set $H \in \mathcal{T}$, can we talk about some open set corresponding to H in the topology $\mathcal{T}'$? The answer is yes, and we can do it by using another function, that is, the inverse of the function f, that is, a function from $(Y, \mathcal{T}')$ to $(X, \mathcal{T})$, and it will exist. If we are computing $(f^{-1})^{-1}(H)$, that will be of H. So, this will be nothing but this f. So, what is our requirement? Our requirement is that the image of this H under inverse of inverse of f should be open. Because the map is bijective, we can take it as $f(H)$. So, the idea is that whenever we are talking about preserving the topological structures, we are taking a bijective function f plus continuity of f plus one more thing we require, which is continuity of f^{-1}. So, we require bijective functions plus bi-continuity, that is, continuity of f along with continuity of f^{-1}.

Again, a question arises: If a function f is continuous, can we deduce something about the continuity of f^{-1}? Note that this is independent from the continuity of f because if we are taking a function $f : (\mathbb{R}, \mathcal{T}_l) \to (\mathbb{R}, \mathcal{T}_e)$ such that $f(x) = x$, for all $x \in \mathbb{R}$, that is the identity function. This function

is always continuous. We already have seen it. But if we are taking a set $G = [2, 3) \in \mathcal{T}_l$, and if we are computing $(f^{-1})^{-1}(G)$, that is, if we are computing $f(G)$, that will be the G itself, which is not a member of $\mathcal{T}_e$. It means that this function f is continuous, but the inverse of this function is not continuous, and therefore, both the concepts are independent.

With these ideas in mind, let us see the formal definition of a homeomorphism. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces. Then a homeomorphism between $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ is a function $f : X \to Y$ such that

- f is bijective;

- f is continuous; and

- f^{-1} is continuous.

If there exists a homeomorphism between $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$; the spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are said to be homeomorphic and is denoted by $(X, \mathcal{T}) \cong (Y, \mathcal{T}')$.

Let us take some of the examples. Begin with, let us take $X = \{a, b, c\}$, and let us take another set $Y = \{p, q, r\}$. Also, let us take topology $\mathcal{T} = \{\emptyset, X, \{a\}, \{b, c\}\}$ on X, and topology $\mathcal{T}' = \{\emptyset, Y, \{p\}, \{q, r\}\}$ on Y. Now, let us define a function $f : X \to Y$ such that $f(a) = p, f(b) = q, f(c) = r$. It is clear that this f is a bijective function. Also, the inverse image of empty set is empty set, the inverse image of Y is X. If we are computing inverse image of this singleton set $\{p\}$, that is a singleton set $\{a\}$, and the inverse image of the set $\{q, r\}$ is the set $\{b, c\}$. It means that f is continuous. Also, if we are computing or if we are verifying the continuity of f^{-1}, $f(\emptyset) = \emptyset$, $f(X) = Y$, $f(\{a\}) = \{p\}$, and $f(\{b, c\}) = \{q, r\}$. Thus, f^{-1} is also continuous, and therefore, this f is a homeomorphism. Now, one thing is clear from here that f is a bijective function, and there is bijectiveness between open sets. What is going on, that is f is sending the empty set to the empty set, and inverse of the empty set is the empty set, $f(X) = Y$ and $f^{-1}(Y) = X$, $f(\{a\}) = \{p\}$ and $f^{-1}(\{p\}) = \{a\}$; and $f(\{b, c\}) = \{q, r\}$ and $f^{-1}(\{q, r\}) = \{b, c\}$. Thus, there is a concept of preserving the open sets, which we were discussing.

Let us take one more example, the well-known example, which is known as the Sierpinski space. Let us take a set $X = \{a, b\}$, take one topology $\mathcal{T}_a = \{\emptyset, \{a\}, X\}$ on it. Let us take another topology $\mathcal{T}_b = \{\emptyset, \{b\}, X\}$. Now,

if we are defining a function $f : X \to X$ such that $f(a) = b$ and $f(b) = a$. Then this function is bijective. Also, if we are computing the image of empty set, image of singleton set $\{a\}$, or image of X, that is, respectively empty set, singleton set $\{b\}$, and X itself, and the same happens when we are talking about the inverse image of empty set, singleton set $\{b\}$ or X. This is similar to the previous example. What is going on here because f is bijective, and f and f^{-1} both are continuous, therefore this is also a homeomorphism.

Moving ahead, let us take one more example. Let us take a function $f : (\mathbb{R}, \mathcal{T}_e) \to (\mathbb{R}, \mathcal{T}_e)$ such that $f(x) = 3x$, for all $x \in \mathbb{R}$. It is clear from the definition itself that this function is bijective. Even if we want to check it, if we are taking $f(x_1) = f(x_2)$, it means that $3x_1 = 3x_2$, or $x_1 = x_2$. Therefore, the function f is injective. Also, for all $y \in \mathbb{R}$, there is a real number $y/3 \in \mathbb{R}$ such that $f(y/3) = y$. Therefore, f is surjective. Also, from calculus, it is clear that this is a linear function and this function is continuous. If we are looking for the inverse of this function, that is $x/3$, which is also continuous. Thus, this function f is bijective, continuous, and f^{-1} is continuous, too. Therefore, this f is a homeomorphism.

Moving ahead, let us discuss this interesting theorem, which provides a relationship between homeomorphic spaces. The statement of the theorem is: Being homeomorphic is an equivalence relation on the collection of all topological spaces. If we want to prove this theorem, let us take this C as a collection of all topological spaces. Now, define a relation R on C. This relation is defined as: two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are related to each other if and only if $(X, \mathcal{T}) \cong (Y, \mathcal{T}')$. Our motive is to justify that this relation is an equivalence relation. For which, let us first see that this relation R is a reflexive relation, which is simple to justify. As for every topological space $(X, \mathcal{T})$, $(X, \mathcal{T}) \cong (X, \mathcal{T})$ because there is a function that we call the identity function from $(X, \mathcal{T})$ to itself, which is a bijective function. Also, the identity function is continuous, and its inverse is also continuous because we have the same topology on the domain as well as co-domain. Thus, $(X, \mathcal{T})$ is always homeomorphic to itself, that is $(X, \mathcal{T})$ is related to itself, and therefore this relation is reflexive.

Moving ahead, let us show that R is symmetric. It is also simple to justify because if $(X, \mathcal{T})$ is related to $(Y, \mathcal{T}')$, it means that this $(X, \mathcal{T}) \cong (Y, \mathcal{T}')$,

or there exists a homeomorphism $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$. Now, if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a homeomorphism, we can talk about another homeomorphism, that is $f^{-1} : (Y, \mathcal{T}') \to (X, \mathcal{T})$. Note that this f is bijective; because of the bijectiveness of f, this f^{-1} is bijective, too. Also, if f is continuous, we conclude that the inverse of f^{-1} is continuous. Being homeomorphism, f^{-1} is continuous, which we also require. Therefore, this f^{-1} is a homeomorphism.

Finally, let us justify that this relation R is transitive. In order to justify it, we take three topological spaces; one is $(X, \mathcal{T})$, which is related to $(Y, \mathcal{T}')$, and let us take this topological space $(Y, \mathcal{T}')$ be related to another topological space $(Z, \mathcal{T}'')$. If these are related to each other, it means that this $(X, \mathcal{T}) \cong (Y, \mathcal{T}')$ and this $(Y, \mathcal{T}') \cong (Z, \mathcal{T}'')$, that is there exists a homeomorphism $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ and there exists also a homeomorphism $g : (Y, \mathcal{T}') \to (Z, \mathcal{T}'')$. As we have to justify that this relation R is transitive, our motive is to show that this $(X, \mathcal{T})$ is related to $(Z, \mathcal{T}'')$; that is, we have to find out a homeomorphism from $(X, \mathcal{T})$ to $(Z, \mathcal{T}'')$. From here, it is clear that if we are taking the composition of these two, obviously that is a function $g \circ f : (X, \mathcal{T}) \to (Z, \mathcal{T}'')$. We cam justify that $g \circ f$ is a homeomorphism. Now, we already know as f and g are homeomorphisms, so f and g are bijective, and if two functions are bijective, their composition is also bijective. We have also seen that if two functions are continuous, their composition, is continuous too. Finally, if f^{-1} and g^{-1} are continuous, then their composition, that is, $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$, is continuous. From here, we can conclude that $(g \circ f)^{-1}$, is continuous too. Hence, this $(X, \mathcal{T})$ is related to $(Z, \mathcal{T}'')$, or the relation R is an equivalence relation. This theorem will help us to deduce the homeomorphism between different topological spaces. We have justified that being homeomorphism is a transitive relation, so whenever we want to deduce that two topological spaces are homeomorphic, if we can show that for example if this $(X, \mathcal{T}) \cong (Y, \mathcal{T}')$ and this $(X, \mathcal{T}) \cong (Z, \mathcal{T}'')$, we can deduce that this $(Y, \mathcal{T}') \cong (Z, \mathcal{T}'')$. Further, as this relation is an equivalence relation, we can talk about the equivalence class of a topological space $[(X, \mathcal{T})]$, and is given by $[(X, \mathcal{T})] = \{(Y, \mathcal{T}') : (X, \mathcal{T}) \cong (Y, \mathcal{T}')\}$.

Moving ahead, let us see an example. The closed intervals $X = [0, 1]$ and $Y = [a, b]$ with $a, b \in \mathbb{R}, a < b$, each as a subspace of $(\mathbb{R}, \mathcal{T}_e)$, are homeomorphic. If we want to prove it, let us see a homeomorphism between these two sets. Define a function $f : [0, 1] \to [a, b]$ such that $f(x) = a + (b - a)x$,

$x \in [0, 1]$. It can be seen from here, that this f is an injective function. Also, this f is surjective because if we are taking any $y \in [a, b]$, there exists $x = (y - a)/(b - a) \in [0, 1]$, such that $f(x) = y$. Therefore, f is surjective. Further, this f is a linear function. Therefore, this f is continuous. Also, if we are computing the inverse of this f, so $f^{-1}(x) = (x - a)/(b - a)$, and that is again a linear function of x, that is continuous. Therefore, this f is a homeomorphism between closed interval $[0, 1]$ and closed interval $[a, b]$, i.e., $[0, 1] \cong [a, b]$. Similarly, we can justify that $[0, 1] \cong [c, d]$, for some other real numbers c and $d, c < d$. Now, by using the concept which we have studied earlier regarding the equivalence relation, we can deduce that $[a, b] \cong [c, d]$, that is, any two closed intervals, when they are taken as subspaces of the set of real numbers along with Euclidean topology are homeomorphic to each other.

Let us take some more examples. This is regarding the homeomorphism between open intervals. We can justify that $(0, 1) \cong (a, b)$, where $a < b$. In order to justify it, define a function which we have already seen. This is a function $f : (0, 1) \to (a, b)$ such that $f(x) = a + (b - a)x$. Then as we have already seen this function is bijective continuous as well as its inverse is also continuous, this f is a homeomorphism between this open interval $(0, 1)$ and open interval (a, b).

Moving ahead, we can justify that $(0, 1) \cong (1, \infty)$. In order to justify it, we can define a function $f : (0, 1) \to (1, \infty)$ such that $f(x) = 1/x$, for all $x \in (0, 1)$. It is clear that this function is bijective. Also, this function is continuous and if we are talking about the inverse of this function that is $1/x$, for all $x \in (1, \infty)$, that is continuous too. Therefore, $(0, 1) \cong (1, \infty)$.

Even if we are taking a as any real number, we can justify that $(-\infty, a) \cong (-a, \infty)$. In order to justify it, we can define a homeomorphism $f : (-\infty, a) \to (-a, \infty)$ such that $f(x) = -x$. It is clear that this f is bijective, continuous and even its inverse is continuous. Therefore, $(-\infty, a) \cong (-a, \infty)$. Even one more thing is here that $(-\pi/2, \pi/2) \cong \mathbb{R}$. To justify it, let us define $f : (-\pi/2, \pi/2) \to \mathbb{R}$ such that $f(x) = tan x$. If we are talking about the inverse of this, that is nothing but $tan^{-1} x$. From calculus, we know that f and inverse of f are continuous, and f is bijective. Therefore, $(-\pi/2, \pi/2) \cong \mathbb{R}$. Finally, what can we conclude? We have seen that $(-\pi/2, \pi/2) \cong \mathbb{R}$. Also, we have seen that this $(0, 1) \cong (a, b), a, b \in \mathbb{R}$. From these two, we can conclude

that $(-\pi/2, \pi/2) \cong (a, b)$. But note that $(-\pi/2, \pi/2) \cong \mathbb{R}$. Therefore, we can conclude that $(a, b) \cong \mathbb{R}$.

These are the references.

That's all from this lecture. Thank you.