Welcome to Lecture 30 on Essentials of Topology.

In this lecture too, we will continue the study of continuous functions. Specifically, we will study the concept of continuous functions associated with relative topology, quotient topology, and product topology. Before starting to study these concepts, let us see a concept that is associated with the concept studied in the previous lecture, that is, a characterization of continuous function in terms of closure. Actually, in the last lecture, we saw a characterization of continuous functions, that is, if f is a function from topological space $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$, we say that this function is continuous if and only if $f(\bar{A}) \subseteq \overline{(f(A))}$, for all $A \subseteq X$. A natural question is, can we characterize the continuity in terms of inverse image as well as closure? The answer is yes, we can do it.

What exactly we will do is we will take some subset B of Y, and we will try to take the inverse image of this B and use the concept of closure to characterize the continuity of this function. The statement of this theorem is as given: if we are having two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$, then the function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, for all $B \subseteq Y$. In order to prove this theorem, let us first assume that f is continuous. Now, let us begin with $B \subseteq Y$. We know that $B \subseteq \bar{B}$, and therefore, what about this $f^{-1}(B)$, that will be a subset of $f^{-1}(\bar{B})$. Now, by taking the closure, we can conclude that $\overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\bar{B})}$, as we already know that if $P \subseteq Q$, $\bar{P} \subseteq \bar{Q}$. Now, one more thing is here, that is this $B \subseteq Y$, what about this $\bar{B}$? Note that this is $\mathcal{T}'$-closed. If this is $\mathcal{T}'$-closed, as f is continuous, $f^{-1}(\bar{B})$ is $\mathcal{T}$-closed. Therefore, by the concept of closure, $\overline{f^{-1}(\bar{B})} = f^{-1}(\bar{B})$. Therefore, we can write $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, and that was our requirement. So, this is the proof of the first part.

Moving ahead, let us assume that for all $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; then, we have to prove that f is continuous. In order to prove it, see the assumption which is given in terms of closure. Therefore, it will be better to use the charac-

terization of continuous maps in terms of closed sets to justify it. Accordingly, let us take a subset F of Y, which is $\mathcal{T}'$-closed. If this is $\mathcal{T}'$-closed, we can conclude that $\bar{F} = F$. In order to justify the continuity of f, what do we have to prove? Actually, we have to justify that $f^{-1}(F)$ is $\mathcal{T}$-closed. In order to justify it, we have to prove that $\overline{f^{-1}(F)} = f^{-1}(F)$, or this is equivalent to two statements. The first one is that $f^{-1}(F) \subseteq \overline{f^{-1}(F)}$, and the second is that $\overline{f^{-1}(F)} \subseteq f^{-1}(F)$. Note that first follows from the definition of closure. So, what exactly do we have to justify? We have to justify only the second one. In order to justify it, now come to the assumption and replace this B with F. So, from here, we can write $\overline{f^{-1}(F)} \subseteq f^{-1}(\bar{F})$, but note that $\bar{F} = F$. So, this can be written as $\overline{f^{-1}(F)} \subseteq f^{-1}(F)$. Therefore, this second also holds, or $f^{-1}(F)$ is $\mathcal{T}$-closed, that is the inverse image of a $\mathcal{T}'$-closed set is a $\mathcal{T}$-closed set. Therefore, f is continuous.

Moving to the next, let us see the continuous function from a subspace to a topological space, and that function is precisely the inclusion function. Let $(A, \mathcal{T}_A)$ be a subspace of $(X, \mathcal{T})$, then what can we prove? We can prove that the map $i : (A, \mathcal{T}_A) \to (X, \mathcal{T})$ is continuous. Note that this function i is from A to X, which is defined as $i(x) = x$, for all $x \in A$. Now, if we want to prove that this function is continuous, let us take $G \in \mathcal{T}$ and try to compute $i^{-1}(G)$. Note that $i^{-1}(G) = \{x \in A : i(x) \in G\} = \{x \in A : x \in G\} = A \cap G$. But note that $A \cap G \in \mathcal{T}_A$. Thus, the inverse image of each $\mathcal{T}$-open set is a $\mathcal{T}_A$-open set, and therefore, this map i is continuous.

Moving to the next one, we have seen that this inclusion function i is continuous when the topology on the domain is the relative topology. Note that we can also justify that the relative topology is the smallest topology on A, which makes this inclusion function continuous. In order to justify it, let us take $i : (A, \mathcal{T}') \to (X, \mathcal{T})$, where this $\mathcal{T}'$ is any arbitrary topology on A, provided this i is continuous. Now, if we want to justify that the relative topology $\mathcal{T}_A$ is the smallest topology, we have to prove that this $\mathcal{T}_A$ is coarser than $\mathcal{T}'$. In order to justify, let us see if we are taking any $G \in \mathcal{T}$, we have already seen that $i^{-1}(G) = A \cap G$. Because this function i is a continuous function, therefore this $A \cap G \in \mathcal{T}'$. But at the same time, it is to be noted that for all $G \in \mathcal{T}$, this $A \cap G \in \mathcal{T}_A$. Thus, what we can conclude from here, that is for all H, which is $A \cap G \in \mathcal{T}_A$, $H \in \mathcal{T}'$, or this $\mathcal{T}_A$ is a topology that is coarser than $\mathcal{T}'$. Thus $\mathcal{T}_A$ is the smallest topology on A, which makes this inclusion

function continuous.

Moving to the next concept, let us discuss the composition of continuous functions. We can justify that the composition of two continuous functions is a continuous function. In order to justify it, let us take a continuous function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ and let us take another continuous function $g : (Y, \mathcal{T}') \to (Z, \mathcal{T}'')$. What we have to deduce is that we have to show that their composition, that is, $g \circ f : (X, \mathcal{T}) \to (Z, \mathcal{T}'')$, is also a continuous function. Now, in order to justify it, let us take $G \in \mathcal{T}''$. Then $(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$. As, g is continuous, $g^{-1}(G) \in \mathcal{T}'$. Further, from the continuity of f, $f^{-1}(g^{-1}(G)) \in \mathcal{T}$. It means that the inverse image of every $\mathcal{T}''$-open set under $g \circ f$ is $\mathcal{T}$-open, and therefore, this composition is continuous.

Moving ahead, again coming to the concept of subspaces, let us take a subspace $(A, \mathcal{T}_A)$ of $(X, \mathcal{T})$ and take a continuous function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$. Then, the restricted function $f|_A : (A, \mathcal{T}_A) \to (Y, \mathcal{T}')$, is also a continuous function. Note that this function is defined as $f|_A(x) = f(x)$, for all $x \in A$. In order to prove that this is a continuous function, let us see an interesting thing here. This is already given that $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a continuous function. So, f function is from X to Y. Also, we can define an inclusion function i from A to X as $A \subseteq X$. So, their composition is $f \circ i$, and if we are trying to find out $(f \circ i)(x)$, this will be nothing but $f(i(x)) = f(x) = f|_A(x)$. But note that f and i are continuous functions. Therefore, their composition, too. Hence, $f|_A$ is a continuous function.

Moving to the next concept, which is the concept of a quotient map. We have already seen that if a topological space $(X, \mathcal{T})$ is given and A is any non-empty set and if we are having a surjective function $q : X \to A$, then by using the topology $\mathcal{T}$ on X, we can put a topology $\mathcal{T}_q$ on A with the help of this quotient map q and this topology is known as quotient topology. Just recall the definition of this topology. This topology was defined as $\{G \subseteq A : q^{-1}(G) \in \mathcal{T}\}$. Now, if we want to justify that this map q is a continuous map, actually it is trivial from the definition of quotient topology itself because if we are taking any $G \in \mathcal{T}_q$, it means that $q^{-1}(G) \in \mathcal{T}$. That is, the inverse image of each $\mathcal{T}_q$-open set is a $\mathcal{T}$-open set. Therefore, q is a continuous map. Also, if we want to see the nature of this quotient topology in terms of its size, then actually,

this is the largest or finest topology on A for which the quotient map q is continuous. In order to justify it, let us take this map $q : (X, \mathcal{T}) \to (A, \mathcal{T}')$, where this $\mathcal{T}'$ is a topology on A such that this q is a continuous map. Our interest is to justify that the quotient topology is the largest topology. So, what exactly do we have to prove? We have to justify that this $\mathcal{T}'$ is coarser than $\mathcal{T}_q$. It can be easily justified here because if we are taking any $G \in \mathcal{T}'$, by continuity of this q, this $q^{-1}(G) \in \mathcal{T}$. Note that if $q^{-1}(G) \in \mathcal{T}$, by the definition of quotient topology, $G \in \mathcal{T}_q$. That is, $\mathcal{T}'$ is coarser than $\mathcal{T}_q$, or this $\mathcal{T}_q$, is the largest topology on A, for which the quotient map q is continuous.

Moving to the next concept, let us see the continuous maps associated with the product topology. We have already seen the concept of projection maps. The concept was:
Let $(X_1 \times X_2, \mathcal{T})$ be the product space, where $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are two topological spaces. Then the maps

$$\pi_1 : (X_1 \times X_2, \mathcal{T}) \to (X_1, \mathcal{T}_1) \text{ such that } \pi_1(x_1, x_2) = x_1, \text{ and}$$

$$\pi_2 : (X_1 \times X_2, \mathcal{T}) \to (X_2, \mathcal{T}_2) \text{ such that } \pi_2(x_1, x_2) = x_2$$

are called projection maps. What we can justify is that both maps are continuous. In order to justify it, let us take the case of this π_1 and see if we are taking any $G_1 \in \mathcal{T}_1$. Let us compute the inverse image of this G_1 under π_1. So, $\pi_1^{-1}(G_1) = \{(x_1, x_2) : \pi_1(x_1, x_2) \in G_1\} = \{(x_1, x_2) : x_1 \in G_1\} = G_1 \times X_2$. Therefore, π_1 is a continuous map. Similarly, if we are taking any $G_2 \in \mathcal{T}_2$, we can justify that this $\pi_2^{-1}(G_2) = X_1 \times G_2 \in \mathcal{T}$. Therefore, π_2 is also a continuous map.

Moving to the next one, an interesting fact about the product topology is that the product topology is the smallest topology on this $X_1 \times X_2$ for which the projection maps π_1 and π_2 are continuous. In order to justify that this product topology is the smallest topology, let us take a topology or any arbitrary topology $\mathcal{T}'$ on $X_1 \times X_2$ so that the projection maps π_1 and π_2 are continuous. Then we have to justify that the product topology $\mathcal{T}$ is coarser than this $\mathcal{T}'$. Recall the fact which we have already seen that if we are taking this collection as $\mathcal{B}' = \{\pi_1^{-1}(G_1) : G_1 \in \mathcal{T}_1\} \cup \{\pi_2^{-1}(G_2) : G_2 \in \mathcal{T}_2\}$, it is a sub-basis for product topology $\mathcal{T}$. In order to justify that this $\mathcal{T}$ is coarser than $\mathcal{T}'$, the first step which we are going to justify that this $\mathcal{B}' \subseteq \mathcal{T}'$. In order

to justify it, let us take $G \in \mathcal{B}'$. Then G will be either of the form $\pi_1^{-1}(G_1)$ or this G is equal to $\pi_2^{-1}(G_2)$, for some $G_1 \in \mathcal{T}_1$ and $G_2 \in \mathcal{T}_2$. Note that π_1 and π_2 are continuous, so if we are taking any $G_1 \in \mathcal{T}_1$, $\pi_1^{-1}(G_1) \in \mathcal{T}'$, and in the same way, if we are taking any $G_2 \in \mathcal{T}_2$, $\pi_2^{-1}(G_2) \in \mathcal{T}'$. Therefore, in any case, G will always be a member of $\mathcal{T}'$. Hence $\mathcal{B}' \subseteq \mathcal{T}'$. Also, as this $\mathcal{B}'$ is a sub-basis for product topology, we can find a basis from it. Let us take that basis be $\mathcal{B}$. Now, if we are taking any $B \in \mathcal{B}'$, what will happen? This B can be written as a finite intersection of members of sub-basis. But what we have already justified that $\mathcal{B}' \subseteq \mathcal{T}'$, it means that this B is a finite intersection of members of $\mathcal{T}'$, and if it is so, we can conclude that $B \in \mathcal{T}'$, that is $\mathcal{B} \subseteq \mathcal{T}'$.

Finally, if we want to justify that this product topology $\mathcal{T}$ is coarser than $\mathcal{T}'$. Let us take any $G \in \mathcal{T}$ and $x \in G$. Because $\mathcal{B}$ is a basis for product topology, we can find $B \in \mathcal{B}$ such that $x \in B \subseteq G$. But note that if $B \in B$, $B \in \mathcal{T}'$, because $\mathcal{B} \subseteq \mathcal{T}'$. So, we can conclude from here that for all $x \in G$, there exists a $\mathcal{T}'$-open set B such that $x \in B \subseteq G$. From here, we can say that this G is $\mathcal{T}'$-neighborhood of x, or we can say that G is a $\mathcal{T}'$-neighborhood of each of its points, and if this is neighborhood of each of its points, we can say that $G \in \mathcal{T}'$. Hence, this $\mathcal{T}$ is coarser than $\mathcal{T}'$, or the product topology is the smallest topology for which the projection maps π_1 and π_2 are continuous.

These are the references.

That's all from this lecture. Thank you.