Course Name: Essentials of Topology
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Welcome to lecture 3 on Essentials of Topology. In continuation of the previous lecture on sets and functions, in this lecture, we will continue to study some more notions associated with sets and functions. We will study the following topics in this lecture.
1. Composition of Functions
2. Inverse of a Function
3. Finite and Countable Sets, and finally,
4. Indexed Family of Sets.

Begin with the concept of the composition of functions. Let us see this figure. There is a function $f : X \to Y$ and another function $g : Y \to Z$. Our motive is to construct a function from X to Z. If we see this figure, if x is this element of X, then f sends it to $f(x)$, that is an element of Y; and if we are computing the image of $f(x)$ under g, that goes to $g(f(x))$. So, a natural choice is here if $f : X \to Y$ and $g : Y \to Z$. Then we can construct a function from X to Z, $g \circ f$ such that $(g \circ f)(x) = g(f(x))$.

Formally, let $f : X \to Y$ and $g : Y \to Z$ be functions. Then their composition is again a function $g \circ f : X \to Z$ such that $(g \circ f)(x) = g(f(x))$. Some simple results can be obtained for the composition of functions. These are the list of results. If f and g are two injective functions, their composition is also injective. If f and g are two surjective functions, their composition is also a surjective function. If f and g are bijective, their composition is also a bijective function.

Let us take some of the examples of the composition of functions. So, the first example is: let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ and $g(x) = x + 3$. Now, let us compute $(g \circ f)(x)$, which is $g(f(x))$, that is $g(x^2)$, and that equals $x^2 + 3$. Similarly, for the second example, if we want to compute $(g \circ f)(x)$, that is $g(f(x))$. As $f(x) = x^2 + 3$. So, this becomes $g(x^2 + 3)$. Also, as $g(x) = \sqrt{x}$, therefore, this equals square root of $x^2 + 3$. In both examples, the domain for

the functions is the set of reals. So, we can not only talk about $g \circ f$, but we can also talk about $f \circ g$.

So, if we are computing this composition for the first example, $(f \circ g)(x)$, that equals $f(g(x))$, and $g(x) = x + 3$ here. What does f do? As $f(x) = x^2$, so this equals $(x + 3)^2$. It is to be noted here that $(g \circ f)(x)$ was $x^2 + 3$, while $(f \circ g)(x)$ is $(x+3)^2$. From here, we conclude that $f \circ g \neq g \circ f$, in this example. Similarly, one can check for the second example.

Moving to the next concept, that is the concept of the inverse of a function. The idea is that if we are having a function from a set X to Y, how do we construct a function from Y to X? We will discuss this in detail here. So, begin with a bijective function $f : X \to Y$. Then the inverse of f is a function $g : Y \to X$ such that $f(x) = y \Rightarrow g(y) = x$.

These are some of the notations which we will use. If there exists some g satisfying this criterion that $f(x) = y \Rightarrow g(y) = x$, then g is called the inverse of f and is denoted by f^{-1}. It is simple to deduce that the inverse of the inverse of a function is the same as the function itself. Also, if we want to compute the inverse of the composition of two functions, g and f, then it equals $f^{-1} \circ g^{-1}$. Let us make an analysis of this definition that why the bijectiveness of the function is necessary.

Begin with this diagram. Here, f is a function from X to Y. It can be seen that this function is not one-one but onto. The idea is: if $f(x) = y$, then our motive is to construct a function g such that $g(y) = x$ and g is a function from Y to X. The question is: how do we find the value of the function at this particular element? Because if f of this element is equal to this one and f of this element is equal to the same thing. Whenever we are trying to find the g image of this element, we are getting two distinct elements in X. Meaning is, this is violating the definition of a function. So, such g cannot exist when f is not one-one.

Moving ahead, here again, is a function $f : X \to Y$. It is clear from the figure that this function is one-one but not onto. With the help of this f, if we want to construct a function $g : Y \to X$, then the question is how to define the function g at this element because none of the elements of the domain are

associated with it. Again, there will be a problem in constructing $g : Y \to X$. That's why it is necessary to begin with a bijective function. Coming to this diagram, here is a function $f : X \to Y$. Let us take this function as a bijective function, where $f(a) = p$, $f(b) = q$, and $f(c) = r$. As this function is both one-one as well as onto, that is, the function is bijective; we can construct a function $g : Y \to X$. By using the definition of f, we can define the function g as: $g(p) = a$, $g(q) = b$, and $g(r) = c$. This g is nothing but the inverse of f.

Let us take some more examples. The first one is: If $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(x) = 2x + 3$. Then, we can construct a function $g : \mathbb{R} \to \mathbb{R}$ such that if we are taking $g(x) = (x - 3)/2$, then g is nothing but the inverse of f. Because if $f(x) = 2x + 3$, then if we want to find out $g(2x + 3)$ by using this definition, this can be written as $(2x + 3 - 3)/2$, and that is equal to x. Therefore, g is the inverse of f.

Coming to the second one, $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(x) = x^2$, which is not a bijective function. If we are computing images, we get, $f(-1) = 1$ and $f(1) = 1$. Also, by visualizing the graph of this function, we can see that the images of distinct points are the same, and therefore, this function is not bijective; thus, we cannot talk about the inverse of this function. But if we are making a simple modification in the domain and codomain of the function and now if we are taking the domain and codomain as $[0, \infty)$, such that $f(x) = x^2$, it can be verified that $f(x) = x^2$ is a bijective function. If this is a bijective function, we can talk about the inverse of this function, and that is nothing but $g(x) = \sqrt{x}$.

We can again verify the definition of the inverse of a function, as $f(x) = x^2$, and if we want to find out what is $g(x^2)$, that is the positive square root of x^2, that is nothing but x, and we can say that g is inverse of f. Moving to the next concept, that is the concept of finite and countable sets. We are recalling some definitions related to the finiteness and countability of sets. We are not going into details here.

Begin with the definition of a finite set. We say that a set X is finite if it is an empty set or there exists a bijective function $f : X \to \{1, 2, \ldots, n\}, n \in \mathbb{N}$. For example, if we are taking the set X as a collection of letters from the English alphabet, then we can say that this set is finite because there ex-

ists a bijective function f from this set to the set $\{1, 2, 3, ..., 26\}$, which sends a to 1, b to 2, and so on, and z to 26. So, this is a simple example of a finite set.

A set which is not finite is called an infinite set. For example, the set of natural numbers, the set of integers, and the set of rational numbers are infinite sets. One can find a number of examples because we cannot construct any bijective function from such sets to any set of the form $\{1, 2, 3, ..., n\}$. Continuing, we say that a set X is countable if either it is a finite set or there exists a bijective function $f : X \to \mathbb{N}$. For example, $\mathbb{N}$ itself is a countable set because the identity function from $\mathbb{N}$ to $\mathbb{N}$ is a bijective function.

Also, the set of integers $\mathbb{Z}$, is countable. Even the set of rational numbers is also a countable set. But it can be seen that the set of real numbers is not countable because no such bijective function from $\mathbb{R}$ to $\mathbb{N}$ exist. Let us see a characterization of countable sets, which is given individually in terms of injective function and surjective function. The result is something like this one. The following statements are equivalent: A set X is countable, is equivalent to there exists an injective function $f : X \to \mathbb{N}$, and that is also equivalent to there exists a surjective function $g : \mathbb{N} \to X$.

As I stated, we will not go into detail regarding these concepts. But for the detailed study of finite and countable sets, one may go through NPTEL course on Real Analysis by Prof. S. H. Kulkarni.

Now, coming to the topic of indexed family of sets. What is it? What will be its importance in topology? We will see later. Let us take a collection of sets $\mathcal{A}$ and another set I. We say that this collection of sets $\mathcal{A}$ is an index collection over I if there is a surjective function $f : I \to \mathcal{A}$ (i.e., for every $A \in \mathcal{A}$ there exists $i \in I$ such that $f(i) = A$). We will always write $f(i)$ as A_i.

Let us take some of the examples of an indexed family of sets. Beginning with the first example, let us take n as a natural number. Take a set $A_n = \{-n, 0, n\}$. Then we can show that $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ is an indexed collection over $\mathbb{N}$. For which, we have to show that there exists a surjective function from the set of natural numbers to such a collection. This is simple to justify because, for every $A_n \in \mathcal{A}$, there exists a natural number n such that $f(n) = A_n$, and therefore, we can say that this collection is an indexed collec-

tion over $\mathbb{N}$. Moving ahead, let us take another example. Let n be an integer and $A_n = [n, \infty)$. Then, we can again see that $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ is an indexed collection over $\mathbb{Z}$. In order to visualize the surjectivity between the set of integers to collection $\mathcal{A}$, this is again a simple one as for every $A_n \in \mathcal{A}$ there exists an integer n such that $f(n) = A_n$.

Moving ahead, let us take another example. Let q be a positive rational number, and $A_q = (-q, q)$. Then, $\mathcal{A} = \{A_q : q \in \mathbb{Q}^+\}$ is again an indexed collection over $\mathbb{Q}^+$. If we want to visualize the surjective function from $\mathbb{Q}^+$ to $\mathcal{A}$, this is a simple one, as corresponding to every A_q there exists $q \in \mathbb{Q}^+$ such that $f(q) = A_q$. Let us take the last example regarding the indexed family of sets. Take any real number r and $A_r = \{(r, y) : y \in \mathbb{R}\}$. Then the collection of such sets is also an indexed collection over $\mathbb{R}$. Because if we are taking any $A_r \in \mathcal{A}$, there exists a real number r such that $f(r) = A_r$. If we want to visualize the structure of A_r, what does it represent? Let us see. Let us take this: x is equal to r, and y is any real number. The points of the form (r, y) lie on a line passing through $x = r$. So, what exactly this A_r is? A_r is nothing but, a line passing through $x = r$ and parallel to the y-axis.

We will discuss two concepts regarding the indexed family of sets. The first such concept is the concept of union. So, let us take an indexed family of sets $\{A_i : i \in I\}$, $(A_i \subseteq X)$ indexed by a nonempty set I. Then

$$\cup\{A_i : i \in I\} = \{x \in X : \exists i \in I \, such \, that \, x \in A_i\}.$$

If we are taking this example, in which n is a natural number, $\mathcal{A}$ is a collection of sets A_n, $n \in \mathbb{N}$, where $A_n = \{-n, 0, n\}$. Then, if we want to compute the union of A_n, $n \in \mathbb{N}$, it can be seen that $\cup\{A_n : n \in \mathbb{N}\} = \mathbb{Z}$. Also, just see that every A_1, A_2, A_3, and so on is a subset of $\mathbb{Z}$, and their union is the set $\mathbb{Z}$.

Let us take another example: for $n \in \mathbb{Z}$, if we are taking the indexed collection $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$, where $A_n = [n, \infty)$. Then, it can be easily seen that A_n is a subset of $\mathbb{R}$, and as we want to find out the union of A_n, $n \in \mathbb{Z}$. This is precisely the set of real numbers $\mathbb{R}$. How is it possible? The answer is, for each real number x, we can always find $n \in \mathbb{Z}$ such that $n \leq x$ or $x \in [n, \infty)$, and whenever we are taking their union, we will get the set $\mathbb{R}$.

Coming to the next example: for $q \in \mathbb{Q}^+$, let $\mathcal{A} = \{A_q : q \in \mathbb{Q}^+\}$, where

$A_q = (-q, q)$, and if we are talking about the union of $\{A_q : q \in \mathbb{Q}^+\}$, that is nothing but $\mathbb{R}$ itself. Coming to the last example: for $r \in \mathbb{R}$, let the collection of sets A_r, where $A_r = \{(r, y) : y \in \mathbb{R}\}$. If we want to see, what is the union of $\{A_r : r \in \mathbb{R}\}$? It can be seen that this is nothing but $\mathbb{R}^2$. So, we have discussed a number of examples of the union of an indexed family of sets.

Moving to the next one, let us see the concept of the intersection of an indexed family of sets. This is similar to the concept of intersection of two sets, which we have already studied. So, let us take $\{A_i : i \in I\}$ be a family of sets $(A_i \subseteq X)$ indexed by a nonempty set I. Then,

$$\cap\{A_i : i \in I\} = \{x \in X : \forall i \in I, x \in A_i\}.$$

For example, if we are taking $n \in \mathbb{N}$, $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$, where $A_n = \{-n, 0, n\}$. Then, it can be seen that $\cap\{A_n : n \in \mathbb{N}\} = \{0\}$ because 0 is common to all the sets. The next one is, for an integer n if $A_n = [n, \infty)$. The question is, what about the intersection of $\cap\{A_n : n \in \mathbb{Z}\}$? It can be seen that this will be an empty set. The question is, why? This is because for all $r \in \mathbb{R}$, that is for all real numbers, there exists $n \in \mathbb{Z}$ such that $x < n$; that is, $x \notin [n, \infty)$, and therefore, x cannot lie in $\cap\{A_n : n \in \mathbb{Z}\}$. What about $\cap\{A_r : r \in \mathbb{R}\}$? where $r \in \mathbb{R}$ and $A_r = \{(r, y) : y \in \mathbb{R}\}$. Just think it.

These are the references.

That's all from today's lecture. Thank you.