Course Name: Essentials of Topology
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Week: 05
Lecture: 05

Welcome to Lecture 29 on Essentials of Topology.

We will continue the study of continuous functions in this lecture. In the previous lecture, we saw the characterization of a continuous function in terms of inverse images of open sets. The question arises whether continuity can be characterized by using the concept of closed sets, neighborhoods, and bases. The answer is yes. We can do it, and that will be discussed in this lecture.

Begin with the characterization of a continuous function in terms of a closed sets. The characterization is given by this theorem. This theorem states that if we are having two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$, a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if for each $\mathcal{T}'$-closed subset F of Y, $f^{-1}(F)$ is a $\mathcal{T}$-closed subset of X. Let us see the proof of this theorem. So, we are assuming that f is a continuous function, and let us try to justify that for each $\mathcal{T}'$-closed subset F of Y, $f^{-1}(F)$ is a $\mathcal{T}$-closed subset of X. So, if we are taking a $\mathcal{T}'$-closed subset F of Y, what about the complement of F in Y? This is a $\mathcal{T}'$-open set. If this is a $\mathcal{T}'$-open set, by continuity of f, $f^{-1}(Y - F)$ is a $\mathcal{T}$-open set. But note that $f^{-1}(Y - F) = X - f^{-1}(F)$, which is a $\mathcal{T}$-open set. If this is open with respect to topology $\mathcal{T}$, it means that $f^{-1}(F)$ is a $\mathcal{T}$-closed set. So, this is the justification of one part of this theorem.

Moving ahead, let us assume that for each $\mathcal{T}'$-closed subset F of Y, $f^{-1}(F)$ is a $\mathcal{T}$-closed subset of X. Then we have to justify that f is a continuous function. So, in order to justify that f is continuous, let us take a $\mathcal{T}'$-open set $G \subseteq Y$. Our motive is to justify that $f^{-1}(G)$ is $\mathcal{T}$-open. Now, if $G \in \mathcal{T}'$, its complement, $Y - G$ is $\mathcal{T}'$-closed. But this is given that the inverse image of each $\mathcal{T}'$-closed subset of Y is a $\mathcal{T}$-closed subset of X, therefore $f^{-1}(Y - G)$ is $\mathcal{T}$-closed. Note that $f^{-1}(Y - G)$ is nothing but $X - f^{-1}(G)$, which is $\mathcal{T}$-closed, or from here, we can deduce that $f^{-1}(G)$ is $\mathcal{T}$-open. Therefore, the function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous.

Moving to the next, let us see the characterization of continuous function in terms of neighborhoods. The characterization is given in terms of this theorem, which states that if we are having two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$, then a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if for each $x \in X$ and each $\mathcal{T}'$-nbd N of $f(x)$, there exists a $\mathcal{T}$-nbd M of x such that $f(M) \subseteq N$. In order to prove this result, first we are assuming that f is a continuous function. Now, let us take a $\mathcal{T}'$- neighborhood N of $f(x)$. By using the definition of neighborhood, we can say that there exists a $\mathcal{T}'$-open set, let us take that is H such that $f(x) \in H \subseteq N$. From here, we can conclude that $x \in f^{-1}(H) \subseteq f^{-1}(N)$. Now, as f is continuous, $f^{-1}(H)$ is a $\mathcal{T}$-open set, or from here, we can say that $f^{-1}(N)$ is a $\mathcal{T}$-neighborhood of x. If this is a $\mathcal{T}$-neighborhood of x, why not let us take this M as $f^{-1}(N)$. So, we have a $\mathcal{T}$-neighborhood of x, and now, if we are computing $f(M)$, which will be equal to $f(f^{-1}(N))$, and that is a subset of N. That's the proof of this part.

Moving ahead, now let us assume that for each $x \in X$ and each $\mathcal{T}'$-neighborhood N of $f(x)$, there exists a $\mathcal{T}$-neighborhood M of x such that $f(M) \subseteq N$, then try to justify that f is a continuous function. In order to justify that f is a continuous function, let us take a $\mathcal{T}'$-open set. So, let G be $\mathcal{T}'$-open, and our motive is to justify that $f^{-1}(G)$ is $\mathcal{T}$-open. Now, if $f^{-1}(G)$ is an empty set, obviously this is a member of topology. In case $f^{-1}(G)$ is a nonempty set, then let us take $x \in f^{-1}(G)$, it implies that $f(x) \in G$, but note that what this G is? G is a $\mathcal{T}'$-open set and therefore this is a neighborhood of each of its point, that is this G is $\mathcal{T}'$-neighborhood of $f(x)$. Now, if this is $\mathcal{T}'$-neighborhood of $f(x)$, by our assumption, there exists a $\mathcal{T}$-neighborhood M of x such that $f(M) \subseteq G$. But note that M is a $\mathcal{T}$-neighborhood of x. Therefore, there exists a $\mathcal{T}$-open set, let us take that is G' such that $x \in G' \subseteq M$. Also, $M \subseteq f^{-1}(G)$. Thus, we conclude that $x \in G' \subseteq f^{-1}(G)$, where G' is a $\mathcal{T}$-open set. It means that this $f^{-1}(G)$ is a $\mathcal{T}$-neighborhood of x. Note that x is an arbitrary element of $f^{-1}(G)$, therefore we can conclude from here that $f^{-1}(G)$ is a $\mathcal{T}$-neighborhood of each of its points, and if this is neighborhood of each of its points, we can conclude that this $f^{-1}(G)$ is $\mathcal{T}$-open. It shows that inverse image of every $\mathcal{T}'$-open set is $\mathcal{T}$-open. Therefore, the function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous.

Moving ahead, let us see another characterization of continuous function and this characterization is given in terms of basis. The statement of this theorem is: if we are having two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$, and let us take

$\mathcal{B}'$ be a basis for $(Y, \mathcal{T}')$, then the function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if for each $B \in \mathcal{B}'$, $f^{-1}(B)$ is a $\mathcal{T}$-open subset of X. In order to justify this theorem, if we are assuming the first one, that is, f is continuous, then what will happen? The inverse image of every $\mathcal{T}'$-open set is $\mathcal{T}$-open. Now, if we are taking any $B \in \mathcal{B}'$ then what will happen? This B will also be a member of this topology $\mathcal{T}'$, or we can say that this B is $\mathcal{T}'$-open. Therefore, by the continuity of f, $f^{-1}(B)$ is $\mathcal{T}$-open, that is the proof of this part.

Now, let us prove the converse part of the result. For that, we are assuming that for each $B \in \mathcal{B}'$, $f^{-1}(B)$ is a $\mathcal{T}$-open subset of X, and try to justify that this function f is continuous. Now, if we are taking any $G \in \mathcal{T}'$, then by the definition of basis, this G can be expressed $\cup\{B_i : i \in I\}$, where $Bi \in \mathcal{B}$. Now, $f^{-1}(G) = f^{-1}(\cup\{B_i : i \in I\}) = \cup\{f^{-1}(B_i) : i \in I\}$. But note that it is already given to us that for $B \in \mathcal{B}'$, $f^{-1}(B)$ is a $\mathcal{T}$-open set. It means that $f^{-1}(G)$ is an arbitrary union of $\mathcal{T}$-open sets. Therefore, this $f^{-1}(G)$ is $\mathcal{T}$-open. Thus, the function f is continuous.

By using this characterization let us see some of the examples. The first example is: Let $f : (\mathbb{R}, \mathcal{T}_e) \to (\mathbb{R}, \mathcal{T}_e)$ be a function such that $f(x) = x + 2$. Then f is continuous. If we want to justify it, we know that for $(\mathbb{R}, \mathcal{T}_e)$ the basis is given by $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$. Now, if we are taking any member of this basis, that is $(a, b) \in \mathcal{B}$, $f^{-1}((a, b)) = (a - 2, b - 2) \in \mathcal{T}_e$. Therefore, by using the previous characterization of a continuous functions, we can say that this f is a continuous function.

Let us take another example. Let $f : (\mathbb{R}, \mathcal{T}_e) \to (\mathbb{R}, \mathcal{T}_e)$ be a function such that $f(x) = 2x$. Then f is continuous. If we are taking $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$, this is a basis for $(\mathbb{R}, \mathcal{T}_e)$. Now, $f^{-1}((a, b)) = (a/2, b/2) \in \mathcal{T}_e$, therefore, this function f is continuous. With the same characterization if we are taking identity function, i.e., let $f : (\mathbb{R}, \mathcal{T}_l) \to (\mathbb{R}, \mathcal{T}_e)$ such that $f(x) = x$. Then f is a continuous function. The justification is similar to the previous one, that is if we are taking the basis $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$, then $f^{-1}((a, b)) = (a, b) \in \mathcal{T}_l$.

If we are looking at this example, what is the relationship between the Euclidean topology and the lower limit topology? We know that this lower limit topology is finer than Euclidean topology. The question is regarding the continuity of identity function. Can we make a characterization by putting a topology on a domain, which is finer than the topology on the co-domain on the function? The answer is yes. We can do and this is a simple characterization of continuous function. When the function is taken as an identity function, that is if $(X, \mathcal{T})$ is a topological space, then $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ such that $f(x) = x, x \in X$ is continuous if and only if $\mathcal{T}$ is finer than $\mathcal{T}'$. The justification is simple. For example, if we are assuming that f is continuous. Now, if we are taking any $G \in \mathcal{T}'$, then $f^{-1}(G) = G \in \mathcal{T}$, or we can say that $\mathcal{T}$ is finer than $\mathcal{T}'$. For the converse part, let us assume that $\mathcal{T}$ is finer than $\mathcal{T}'$. Now, if we want to justify that f is continuous, let us take again $G \in \mathcal{T}'$. Then by using our assumption, G is a member of $\mathcal{T}$. But what G exactly is? G is nothing but $f^{-1}(G)$. Thus f is a continuous function.

Moving ahead, let us see how a continuous function can be characterized by using the concept of closure. The statement of the theorem is given as: Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two topological spaces. Then $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous iff $f(\overline{A}) \subseteq \overline{f(A)}$, for all $A \subseteq X$. Let us assume that this function is continuous and justify that $f(\overline{A}) \subseteq \overline{f(A)}$, for all $A \subseteq X$. In order to prove it, this is given to us that A is a subset of X. Therefore, $f(A) \subseteq Y$, or $\overline{f(A)} \subseteq Y$ is $\mathcal{T}'$-closed. Now, if this is $\mathcal{T}'$-closed because this function f is continuous, we can say that $f^{-1}(\overline{f(A)})$ is $\mathcal{T}$-closed. If this is $\mathcal{T}$-closed, what can we conclude? The closure of $f^{-1}(\overline{f(A)})$ will always be equal to $f^{-1}(\overline{f(A)})$. Now, from the definition of closure, we can write $f(A) \subseteq \overline{f(A)}$, or $f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$. We know that $A \subseteq f^{-1}(f(A))$. Therefore, $A \subseteq f^{-1}(\overline{f(A)})$, or $\bar{A}$ is a subset of closure of $f^{-1}(\overline{f(A)})$. But we have already seen that the closure of $f^{-1}(\overline{f(A)})$ is the same as $f^{-1}(\overline{f(A)})$, so, $\bar{A}$ is a subset of $f^{-1}(\overline{f(A)})$, i.e., $f(\overline{A}) \subseteq \overline{f(A)}$.

Moving to the converse part of this result. Now, let us assume that $f(\overline{A}) \subseteq \overline{f(A)}$, for all $A \subseteq X$. Then we have to justify that f is a continuous function. In order to justify that f is a continuous function because the assumption is given in terms of closure, it is always better to use the characterization of continuous function in terms of a closed set. So, let us take any $F \subseteq Y$, which is $\mathcal{T}'$-closed, and to show the continuity, what we have to justify, that this $f^{-1}(F)$ is $\mathcal{T}$-closed. Now, if we want to justify that this is $\mathcal{T}$-closed it is equivalent to show that closure of $f^{-1}(F)$ is equal to $f^{-1}(F)$, or it is equivalent to show two things: one is $f^{-1}(F) \subseteq \overline{f^{-1}(F)}$, and the second thing is try to justify that $\overline{f^{-1}(F)} \subseteq f^{-1}(F)$. The first one follows from the definition of closure. Therefore, we need to only justify that $\overline{f^{-1}(F)} \subseteq f^{-1}(F)$. In order to justify this part, let us use the assumption which is given to us. Note that $f^{-1}(F) \subseteq X$. If we are assuming that $f^{-1}(F)$ is A, then by using this assumption, $f(\overline{f^{-1}(F)}) \subseteq \overline{f(f^{-1}(F))} \subseteq \overline{F} = F$. So, what have we got? We have with us $f(\overline{f^{-1}(F)}) \subseteq F$, or that $\overline{f^{-1}(F)} \subseteq f^{-1}(F)$. That's the proof, therefore f is a continuous function.

These are the references.

That's all from this lecture. Thank you.