Course Name: Essentials of Topology
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Welcome to Lecture 28 on Essentials of Topology.

From this lecture, we will initiate the study of continuous functions. Begin with the concept of continuous function defined over the set of real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and a be a real number. We are familiar that f is continuous at $x = a$, if for given $\epsilon > 0$ there exists a real number $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. This is the well-known definition of continuity and is known as an $\epsilon - \delta$ definition. If we analyze this definition, from here, we can say that $a - \delta < x < a + \delta$, and it implies that $f(a) - \epsilon < f(x) < f(a) + \epsilon$. This condition can be visualized as: if this is the real line, let us take the interval as $a - \delta$ and $a + \delta$. Also, if we are taking $f(a)$ here, then this is the interval which is containing $f(a)$ given by $(f(a) - \epsilon, f(a) + \epsilon)$. Specifically, what we can write that if $x \in (a - \delta, a + \delta)$, it implies that $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$.

The question that comes from this definition of continuity is, how can we define the notion of continuous functions in topological spaces? As in topological spaces, we take X as an arbitrary non-empty set; we cannot talk about the intervals that we have taken here. So, what to do? The idea is if we are replacing this $(f(a) - \epsilon, f(a) + \epsilon)$, for example, we are considering this as a set H. Note that $(f(a) - \epsilon, f(a) + \epsilon)$ is the open interval, and open intervals are open sets in $\mathbb{R}$. So, we are saying that H itself is an open set, and this open set is containing $f(x)$. Let us take the interval $(a - \delta, a + \delta)$ as an open set G. This open set contains x. Now, by using this definition, we can conclude that $f(G) \subseteq H$. So, from here, what we can deduce is that if we are taking any open set H containing $f(x)$, there exists an open set G containing x such that $f(G) \subseteq H$. Actually, this is the basic definition of the continuity of a function at a point a. Let us visualize it. What we have seen if we are taking some arbitrary sets, let us take this one is a set X and another set is Y. If we are defining a function $f : X \to Y$ and we want to check the continuity at an arbitrary point of the domain, let us take that point as x_0 instead of a. What we have to do is that this f is continuous at x_0 if we are finding $f(x_0)$, take any open set H containing this $f(x_0)$. Corresponding to it, there exists an open set G here containing x_0 such that $f(G)$ lies inside H. This is the formal definition of a continuous function at a point x_0. So, formally, if we are having a topological space $(X, \mathcal{T})$ and another topological space $(Y, \mathcal{T}')$, and we are taking a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$, that is precisely a function $f : X \to Y$, and let us take a point $x_0 \in X$. We say that f is continuous at x_0 if for each $\mathcal{T}'$-open set H containing $f(x_0)$, there exists a $\mathcal{T}$-open set G such that $f(G) \subseteq H$.

Let us see some of the examples. For example, if we are taking a set $X = \{a, b, c\}$. Also, let us take another set $Y = \{p, q, r\}$. Define a map $f : X \to Y$ such that $f(a) = p, f(b) = r$ and $f(c) = q$. Now, if we are taking topology $\mathcal{T}$ on X as $\mathcal{T} = \{\emptyset, X, \{a\}, \{b, c\}\}$. Also, let us take a topology $\mathcal{T}'$ on Y, that is $\{\emptyset, Y, \{p\}, \{q, r\}\}$. Now, let us check the continuity at $x = a$. If we are taking $\mathcal{T}'$-open sets which are containing $f(a) = p$, these are either singleton set $\{p\}$ or Y. Now, our motive is to search some $\mathcal{T}$-open set G such that its image is contained in this singleton set $\{p\}$ as well as some $\mathcal{T}$-open set whose image is contained in Y. If we are taking this singleton set $\{a\}$, then it is $\mathcal{T}$-open and $f(\{a\}) \subseteq \{p\}$ and $f(\{a\}) \subseteq Y$. Therefore, we can say that f is continuous at $x = a$. Similarly, if we want to check the continuity of the function at $x = b$, again, the $\mathcal{T}'$-open sets containing this $f(b) = r$ are Y as well as the set $\{q, r\}$. Now, again, our motive is to find out some $\mathcal{T}$-open set whose image is contained in Y and another $\mathcal{T}$-open set, or it may be the same whose image is contained in this set $\{q, r\}$. Why not let us take the $\mathcal{T}$-open set $\{b, c\}$. If we are finding its image, note that its image will be $\{q, r\}$. Note that $f(\{b, c\}) \subseteq \{q, r\}$ and $f(\{b, c\}) \subseteq Y$. Therefore, again, the function is continuous at $x = b$. Similarly, if we are taking $x = c$, we have the similar observations. Thus, this function f is continuous at all the points of the domain; that is, $x = a, b$ as well, as c.

Let us take another example. In this example, we are continuing with the same function, that is, $X = \{a, b, c\}$, $Y = \{p, q, r\}$ and $f : X \to Y$ such that $f(a) = p, f(b) = r$ and $f(c) = q$. Now, if we are making a simple change in the topology, let us take the topology $\mathcal{T}$ on X to be given by $\{\emptyset, X, \{b\}, \{a, c\}\}$. Now, let us take $\mathcal{T}'$, this is $\{\emptyset, Y, \{p\}, \{q, r\}\}$. If we check the continuity at different points, for example let us check the continuity at $x = a$, what about this $f(x)$, that is $f(a)$? Note that this is p. Now, if we are taking $\mathcal{T}'$-open sets containing p, so such open sets are $\{p\}$ as well as Y. Our motive is to search for some $\mathcal{T}$-open sets G and G' containing a so that $f(G) \subseteq \{p\}$ as well as $f(G') \subseteq Y$. Now, the $\mathcal{T}$-open sets containing a are either $\{a, c\}$ or X. If we find the image of $\{a, c\}$ under f, then this image is given by the set $\{p, q\}$. Similarly, $f(X) = Y$. Meaning is, we cannot find any $\mathcal{T}$-open set G here so that $f(G) \subseteq \{p\}$. Therefore, this function is not continuous at $x = a$. Similarly, we can check the continuity at b as well as c. So, from this example, we can conclude that the same function may not be continuous if we are changing the topology.

Moving ahead, we say that a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if it is continuous at each element of the domain.The question arises whenever we want to check the continuity on the domain whether this is the only way to check the continuity at each and every point because this is only the definition that we have seen. Can we simplify it, or can we characterize the continuity in some different and simple manner? The answer is yes, and that is given in the form of this theorem. The statement of the theorem is if we are taking a function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$, this function is continuous if and only if for each $\mathcal{T}'$-open set G, the inverse image of G under f is a $\mathcal{T}$-open set, i.e., for all $G \in \mathcal{T}'$, $f^{-1}(G) \in \mathcal{T}$.

Let us see the proof of this theorem. Begin with by assuming that the function f is continuous. Now, what we have to justify is that for each $\mathcal{T}'$-open set G, show that $f^{-1}(G)$ is a $\mathcal{T}$-open set. If we are computing $f^{-1}(G)$, it may happen that $f^{-1}(G)$ is an empty set, and if this is so, obviously $f^{-1}(G) \in \mathcal{T}$. In case $f^{-1}(G)$ is non-empty, try to prove that $f^{-1}(G)$ is a neighborhood of each of its points. So, let us take an element $x \in f^{-1}(G)$. Then $f(x) \in G$. Note that this function is continuous, so there is a $\mathcal{T}$-open set, let us take that is, G', this is containing x such that $f(G') \subseteq G$. Now, take the inverse image, so $f^{-1}(f(G')) \subseteq f^{-1}(G)$. As we know that, $f^{-1}(f(G'))$ is always a superset of G'. Therefore, $G' \subseteq f^{-1}(G)$. Also, G' is containing x, so we can write $x \in G' \subseteq f^{-1}(G)$. From here, we can conclude that $f^{-1}(G)$ is a neighborhood of x. Because this x is an arbitrary element of $f^{-1}(G)$, so this $f^{-1}(G)$ is a neighborhood of each of its points, and therefore $f^{-1}(G)$ is $\mathcal{T}$-open.

In order to prove the converse part of this theorem, assume that for each $\mathcal{T}'$-open set G, $f^{-1}(G)$ is $\mathcal{T}$-open. Then our motive is to prove that f is continuous. If we want to prove that f is continuous, try to prove that this function is continuous at each point of the domain. So, let us take any $x \in X$. If we want to prove the continuity of the function at this point, let us take a $\mathcal{T}'$-open set H containing $f(x)$. Our motive is to find a $\mathcal{T}$-open set G such that it should contain x, and we have to justify that $f(G) \subseteq H$. Now, if $f(x) \in H$, from here, we can conclude that $x \in f^{-1}(H)$. But note that this is already given that if we are taking any $\mathcal{T}'$-open set H, its inverse image is a $\mathcal{T}$-open set. Therefore, $f^{-1}(H)$ is $\mathcal{T}$-open. Now, if we are taking this $f^{-1}(H)$ as G, and if we are computing the image of G under f, that is nothing but $f(f^{-1}(H)) \subseteq H$. So, we have shown that if we are taking any $\mathcal{T}'$-open set H containing $f(x)$ there exists a $\mathcal{T}$-open set $G \in \mathcal{T}$ so that $f(G) \subseteq H$, therefore f is continuous.

Now, let us discuss some of the examples of continuous as well as non-continuous functions. Begin with the first one, let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$, where $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are two topological spaces, and f is a constant function, then f is always a continuous function. Why not let us define $f(x) = y_0$, where $x \in X$, and y_0 is a fixed element of Y. If we want to find the inverse image of $\mathcal{T}'$-open sets, let us take any $G \in \mathcal{T}'$. There are two possibilities with this G. It may happen that this $y_0 \in G$ or y_0 is not an element of G. In case $y_0 \in G$, $f^{-1}(G) = X$ because $f(x) = y_0$, for all $x \in X$. Now, if y does not belong to G, $f^{-1}(G) = \emptyset$. In any case, $f^{-1}(G)$ is always a member of $\mathcal{T}$, that is the inverse image of every $\mathcal{T}'$-open set is a $\mathcal{T}$-open set. Therefore, this function f is continuous.

Moving to the next one, this is an example of identity function, that is, let $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ be the identity function. Then f is continuous. Note that this function f will always be continuous because if we are taking any $G \in \mathcal{T}$, $f^{-1}(G) = G \in \mathcal{T}$, that is the topology on the domain of the function. Therefore, the identity function is always continuous if we are taking the same topology on the domain as well as co-domain. Moving to the next one, let us take identity function again with a change in the topology, that is, let $f : (\mathbb{R}, \mathcal{T}_e) \to (\mathbb{R}, \mathcal{T}_l)$ be the identity function. Then f is not continuous. We have seen the characterization of continuity, we say that a function is continuous if the inverse image of every $\mathcal{T}'$-open set is a $\mathcal{T}$-open set, but whenever we are just trying to justify that f is not continuous, try to show that there exist some $\mathcal{T}'$-open set whose inverse image is not a $\mathcal{T}$-open set, meaning is if we want to justify that this function is not continuous, let us take some $\mathcal{T}_l$-open set here, that is $G \in \mathcal{T}_l$, and try to justify that $f^{-1}(G)$ is not member of $\mathcal{T}_e$. If we are taking this G as the semi-open interval $[0, 1) \in \mathcal{T}_l$, note that $f^{-1}(G) = [0, 1)$, which is not a member of the Euclidean topology. Therefore, this function is not continuous.

Moving ahead, let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be an arbitrary function and the topology $\mathcal{T}$ is discrete. Then f is continuous. The justification is simple because if we are taking any $\mathcal{T}'$-open set, $f^{-1}(G) \subseteq X$, therefore $f^{-1}(G) \in \mathcal{T}$. So, f is a continuous function. This is an interesting result that any arbitrary function becomes continuous, provided the topology on the domain is discrete.

In the same fashion, if we are putting a restriction on the topology on co-domain, we can construct a family of continuous functions, and that is given in this example. That is here, $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is again an arbitrary function, but note that the topology $\mathcal{T}'$ is an indiscrete topology. In this case, f will always be continuous. The justification is simple because this $\mathcal{T}'$ is an indiscrete topology; there will be only two open sets here, that is, Y and $\emptyset$. Now, $f^{-1}(Y) = X$ and $f^{-1}(\emptyset) = \emptyset$. Meaning is, in any case $f^{-1}(G) \in \mathcal{T}$, for all $G \in \mathcal{T}'$. Therefore, f is continuous.

These are the references.

That's all from this lecture. Thank you.