

**Course Name: Essentials of Topology**  
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Welcome to Lecture 27 on Essentials of Topology.

In this lecture, we will continue the study of the concept of countability axioms. In the previous two lectures, we have studied the concept of first countable spaces and second countable spaces. In this lecture, we will study the concept of separable spaces and try to explore the possible relationship with first countable spaces as well as second countable spaces. Begin with the definition of a separable space. A topological space  $(X, \mathcal{T})$  is called separable if it has a countable dense subset. Meaning is to say that if we are having a topological space  $(X, \mathcal{T})$ , we say that this space is separable if there exists  $A \subseteq X$  so that  $cl(A) = X$ . Note that this subset should be countable.

Let us take some of the examples. The first one is,  $(\mathbb{R}, \mathcal{T}_e)$  is a separable space. This is well known because if we have a real line with Euclidean topology, we know that  $cl(\mathbb{Q}) = \mathbb{R}$ , and the set of rational numbers is countable. Similarly to it, we can also justify that  $\mathbb{R}^2$  with Euclidean topology is separable, and this is because  $cl(\mathbb{Q} \times \mathbb{Q}) = \mathbb{R}^2$ . Even we can generalize this result, and we can justify that  $\mathbb{R}^n$  with the Euclidean topology on it, is also separable. This is because  $cl(\mathbb{Q} \times \mathbb{Q} \times \dots \times \mathbb{Q}) = \mathbb{R}^n$ , where the product is of  $n$ - sets. Therefore,  $\mathbb{R}^n$  with the Euclidean topology is also separable.

Moving to the next example, an uncountable set with the cofinite topology is also separable. The justification behind this example is, let us take this as  $X$  and the cofinite topology on  $X$ . Note that this  $X$  is uncountable. Now, let us take a subset  $A$  of  $X$ . We know that if  $A$  is finite,  $cl(A) = A$ . What about if we are taking a subset  $A$  of  $X$ , which is countable, but this is non-finite? If we compute  $cl(A)$ ,  $cl(A) = X$ . Why? Because  $X$  is the only closed set containing  $A$ , as all other closed sets are finite. Therefore, this uncountable set with cofinite topology is separable.

This is an example of a space that is not separable; that is, an uncountable

set with discrete topology is not separable. Why? Let us take that this  $X$  is an uncountable set, and the topology on it is discrete. We know that if  $A$  is a subset of  $X$ , in discrete spaces,  $cl(A) = A$ . So, what about  $cl(X)$ ?  $cl(X)$  will always be equal to  $X$ , and if  $A$  is a proper subset of  $X$ ,  $cl(A)$  cannot be equal to  $X$ . As  $X$  is the only closed set whose closure is equal to  $X$  but note that  $X$  itself is uncountable, and we cannot find any countable subset of  $X$  whose closure equals to  $X$ . Therefore, uncountable set with discrete topology is not separable.

Moving to the next, let us see the nature of subspaces in the case of separable spaces. We have already seen that subspaces of first and second countable spaces are first as well as second countable. Contrary to these, a subspace of a separable space may not be separable. Let us see this by a counter-example. The topological space which we are taking here, we are taking  $X$  as an uncountable set. Let us take a fixed element  $x_0 \in X$  and define this  $\mathcal{T}$  as a collection  $\{G \subseteq X : G = \emptyset \text{ or } x_0 \in G\}$ . We have already seen this topology, known as Particular Point Topology. If we see this topology, what about the closed sets in this topology? If we are taking a proper subset  $A$  of  $X$ , which is closed, what will happen?  $x_0$  is not an element of  $A$ . But note that  $X$  is also closed. That is, except  $X$ , none of the closed subsets of  $X$  will contain  $x_0$ . Therefore, if we are taking  $A = \{x_0\}$ , and we are computing  $cl(A)$ ,  $cl(A) = X$ . As  $A$  is a singleton set, this is a countable set, and its closure is equal to  $X$ . So, we conclude that this  $(X, \mathcal{T})$  is a separable space. Moving to the next, let us take a subset  $Y$  of  $X$ , and this  $Y$  is  $X - \{x_0\}$ . If we see the nature of relative topology here, note that for all  $y \in Y$ ,  $\{y\} = Y \cap \{x_0, y\}$ . This set  $\{x_0, y\} \in \mathcal{T}$ . It means that for all  $y \in Y$ ,  $\{y\} = Y \cap G$ , where  $G \in \mathcal{T}$ . From here, we can conclude that for all  $y \in Y$ ,  $\{y\} \in \mathcal{T}_Y$ , or that this relative topology is discrete. But what is  $Y$ ? As we have taken  $Y = X - \{x_0\}$ , and  $X$  is uncountable, therefore  $Y$  is uncountable. So, what have we got? We have seen the relative topology on an uncountable set, and the topology is discrete. We have already seen that an uncountable set with discrete topology is not separable. Hence, this particular statement that is a subspace of a separable space may not be separable.

But the question is, if we are putting some restrictions on subspaces, is it possible that subspaces of separable space become separable? The answer is yes. Instead of taking a general subspace, if we are taking an open subspace of

a separable space, then that subspace will always be separable. Let us justify it. If we are taking a topological space  $(X, \mathcal{T})$ , given that this topological space is separable. Now, let us take an open subset  $Y$  of  $X$ , and put the relative topology  $\mathcal{T}_Y$  on it. Our motive is to justify that  $(Y, \mathcal{T}_Y)$  is separable. Note that because this  $(X, \mathcal{T})$  is a separable space, we can find a subset  $A$  of  $X$ , this is countable, so that  $cl(A) = X$ . When we want to show that this  $(Y, \mathcal{T}_Y)$  is separable, we have to find out a countable subset of  $Y$ , so that the closure of that subset equals to  $Y$ . So, how to construct that one? Let us take a set  $B$ , which is precisely  $Y \cap A$ . It is clear from here that  $B \subseteq Y$ , and because we have taken  $B$  as  $Y \cap A$ , this  $B$  is also countable. Now, it remains only to justify that  $cl(B) = Y$ , that is,  $B$  is a dense subset of  $Y$ . Recall the concept of dense sets. If we want to justify that  $cl(B) = Y$ , it is equivalent to justify that for all nonempty  $\mathcal{T}_Y$ -open sets  $H$ , this  $B \cap H$  is non-empty. So, if you want to justify it, let us take  $H \in \mathcal{T}_Y$ ,  $H$  is non-empty. Because  $H \in \mathcal{T}_Y$ , therefore,  $H$  can be expressed as  $Y \cap G$ , where  $G \in \mathcal{T}$ . But what is given to us,  $G \in \mathcal{T}$ , and from our assumption, actually, we are working with open spaces, that is  $Y$  is  $\mathcal{T}$ -open, we conclude that  $H$  is also  $\mathcal{T}$ -open. But note that  $cl(A) = X$ , therefore this  $H \cap A$  is non-empty. From here, we can conclude one more thing that is if we are taking an intersection with  $Y$ , that will also be non-empty, as  $H \subseteq Y$  and  $H$  is non-empty. From here, we can also write  $H \cap Y \cap A$  is non-empty, but  $Y \cap A$  is nothing but  $B$ , therefore  $H \cap B$  is non-empty. That's all. We justified that  $cl(B) = Y$ , meaning is that the open subspace, which is  $(Y, \mathcal{T}_Y)$ , is also separable.

Moving to the next, now we have knowledge about first countable spaces, separable spaces, and second countable spaces. Let us try to explore the possible relationships among these spaces. Beginning with first countable spaces and separable spaces. Our first result is: a first countable space may not be separable. The counter-example for it is a set  $X$  with a topology, this topology is discrete, and the set  $X$  is uncountable. What we have seen is that an uncountable set with the discrete topology is first countable, but this topological space is not separable. Now, if we want to check whether separable spaces are first countable spaces, the answer is again negative. Meaning is, separable spaces may not be first countable. We have already seen an example. The example is, let us take the set  $X$  with cofinite topology, where  $X$  is uncountable. We have seen that this topological space is separable but not first countable. So, the final conclusion is a first countable space may not be separable, and a

separable space may not be the first countable space.

Moving to the second countable spaces and separable spaces. Actually, the result is: Every second countable space is separable. We want to justify this result or this theorem. Let us take a topological space  $(X, \mathcal{T})$ . Given that this topological space is second countable. If this topological space is second countable, it will have or it has a countable basis. Let us take that countable basis to be given by  $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ . When we want to justify that  $(X, \mathcal{T})$  is separable, we have to show that it has a countable dense subset. The question is, how to construct that subset, and that should be countable too. Idea is, we have this basis  $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ , and it is countable. So, from here, if we are constructing a set  $A = \{x_n : x_n \in B_n\}$ . What exactly are we doing? From each element of basis, we are choosing an element, and obviously, this we can do. One thing is clear from here that this  $A$  is countable. Now, the question is whether  $cl(A) = X$ ? The answer is positive, and we can justify. We can show that for all, nonempty set  $G \in \mathcal{T}$ ,  $G \cap A$  will always be non-empty. In order to justify it, let us take  $G \in \mathcal{T}$ , which is non-empty. If we are taking any  $x \in G$ , what will happen? Because  $\mathcal{B}$  is a basis for topology. So, there exists some element there. For example, let us take this  $B_m \in \mathcal{B}$  such that  $x \in B_m \subseteq G$ . Note that from this construction itself,  $A \cap B_m$  is non-empty, for all  $n \in \mathbb{N}$ . Therefore, if  $A \cap B_m$  is non-empty, what we can conclude is  $A \cap B_m \subseteq A \cap G$ , and note that this is non-empty. Therefore,  $A \cap G$  is non-empty or  $cl(A) = X$ . So, every second countable space is separable.

The next one, that is the converse part of this result, and the answer is in the negative, that is, there exists a separable space which may not be second countable. Meaning is, separable spaces may not be second countable. The answer is an uncountable set with cofinite topology; that is, this set is uncountable. We have already seen that this space is separable. Similar to the justification that we have shown in the case of first countable spaces that an uncountable set with cofinite topology is not first countable, we can justify that this space is not a second countable space. Thus, what we finally have with us regarding the relationship among these three spaces. We have studied the concept of first countable spaces, we have also seen the concept of second countable spaces, and finally, we have studied separable spaces. We have seen that every second countable space is a first countable space, but a first countable space may not be second countable. We have seen that separable space

may not be first countable, and also, we have seen that a first countable space may not be separable. We have seen that every second countable space is separable, but there exist separable spaces which may not be second countable. These are all the relationships among these three spaces.

These are the references.

That's all from this lecture. Thank you.