Course Name: Essentials of Topology
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Welcome to Lecture 26 on Essentials of Topology.

In this lecture, we will continue the study of countability axioms. In the previous lecture, we studied the concept of first countable spaces and saw a number of examples. In this lecture, we will study a theorem regarding the convergence of a sequence in first countable spaces. After that, we will study the concept of second countable spaces and will try to explore the relationship between first countable spaces and second countable spaces. As our motive is to discuss a result regarding the convergence of sequences in the first countable space, we will begin with the concept of convergence of a sequence.

In real analysis, we have already studied this concept. Let X be a non-empty set. Then a sequence in X is a function $f : \mathbb{N} \to X$, where $f(n)$ is given as x_n, that is, any term of the sequence, and this is denoted as (x_n), or this is written as $(x_1, x_2, ..., x_n, ...)$. In real analysis, we talk about the convergence of a sequence. So, let us take sequence (x_n) in $\mathbb{R}$. We say that this sequence (x_n) converges to x if, for all $\epsilon > 0$, there exists a natural number n_0 such that $x_n \in (x - \epsilon, x + \epsilon)$, for all $n \geq n_0$. That is if this is our x and this is the open interval centered at x, that is $(x - \epsilon, x + \epsilon)$, this open interval is containing x, then all the elements when $n \geq n_0$ lies in this interval. It means only finite number of elements lie outside this interval. With this definition of convergence of sequence, let us have the concept of convergence of sequence in topological spaces. In a topological space $(X, \mathcal{T})$, if we are taking a sequence (x_n), we say that this sequence (x_n) is converges to an element x of X if for each open set G containing x, that is if we are comparing with real analysis, we are replacing the open interval by an open set, there exists a positive integer n_0 such that $x_n \in G$, for all $n \geq n_0$. What does it mean? Actually, if this is the set X, if this point is x, and if we are taking a sequence (x_n) in this $(X, \mathcal{T})$, if we are taking any open set containing this x, what our motive is or what we have to show that almost all the elements, that is except finite elements, rest of the elements lie in this open set. When (x_n) is converging to x, we say that

x is a limit of the sequence. Also, from the diagram which we have seen here, we can conclude that if the sequence is converging to a point x, it means that if we are taking any open set G containing x, the sequence eventually enters and stays in G.

Let us take some of the examples. We are taking the sequence $x_n = \frac{(-1)^n}{n}$, it converges to 0 in $(\mathbb{R}, \mathcal{T}_e)$. Let us see the justification behind the convergence of this sequence. Let $G \in \mathcal{T}_e$, and this open set is containing 0. Then, from the definition of topology, we can construct an open interval, that is, $(-\epsilon, \epsilon), \subseteq G$, where $\epsilon > 0$. It can be visualized in this form, that is if this is our set of real numbers, this is 0, let us take the open interval like it, this is $-\epsilon$ and this is $+\epsilon$. What is going on here, if we are looking at this sequence, that is $x_n = \frac{(-1)^n}{n}$, then we can conclude that we can find some $n_0 \in \mathbb{N}$ such that $x_n \in (-\epsilon, \epsilon)$, and therefore $x_n \in G$. What is the nature of this sequence? In this sequence, the terms are positive and negative. So, the points will lie in this interval and this n_0 exists because of the Archimedean property. Therefore, this sequence converges to 0 in $(\mathbb{R}, \mathcal{T}_e)$. But if we change the topology on $\mathbb{R}$ and take the lower limit topology $\mathcal{T}_l$, then this sequence will not converge to 0. Why? The answer is, if we are taking an open set $[0, \epsilon)$. Then we cannot find any n_0 so that the sequence lies in this particular open set because, alternatively, the sequence will move outside to this interval. Therefore, this sequence does not converge to 0.

Moving to the next example, let us take $X = \{a, b\}$, and the topology $\mathcal{T}$ on X is given as $\{\emptyset, X, \{a\}$. This topology is known as Sierpensky topology. Now, if we are taking a sequence in this topology, let us take a sequence x_n in X which is a constant sequence and given by $x_n = a$, for all $n \in \mathbb{N}$. Now, this sequence converges to a. Why? Because if we are taking any open set containing a, note that there are only two open sets, i.e., X as well as $\{a\}$, these are the open sets containing a, and $x_n \in X$ and $x_n \in \{a\}$, for all n. Therefore, the sequence is converging to a. Interesting fact about this sequence is that this sequence also converges to b because the open set containing b is only one, that is, X, and $x_n \in X$, for all n. It means that the sequence that we have considered here is converging to two elements, but what we have already studied in real analysis whenever we talk about the convergence of a sequence, we always think that the limit of the sequence is always unique, but in topology what we have seen that the limit of a convergent sequence may not be unique. This is one of the reasons that's why sequences are not much considered in topologies.

Now, coming to the concept of first countable spaces and, let us try to deduce the result of the sequence in first countable spaces. Let us first have a result regarding a first countable space. What this result says that if we are having a first countable space $(X, \mathcal{T})$, then there exists a countable basis $\mathcal{B}_x = \{C_n : n \in \mathbb{N}\}$ at $x \in X$ such that $C_{n+1} \subseteq C_n, \forall n = 1, 2, 3, ...$ Because this $(X, \mathcal{T})$ is already a first countable space, and if this is a first countable space, then there exists a countable basis at each of its elements. So, if we are taking $x \in X$, we can find a countable basis $\{B_n : n \in \mathbb{N}\}$ at x. Now, construct a set C_n, which is given as $\cap\{B_i : 1 \leq i \leq n\}$. If this is the definition of C_n, then it is clear that $C_{n+1} \subseteq C_n$. Now, let us take this $\mathcal{B}_x = \{C_n : n \in N\}$. So, this $\mathcal{B}_x$ has a feature, that is $C_{n+1} \subseteq C_n$, where $n = 1, 2, ...$ Note that from this definition, C_n is the intersection of or finite intersection of B_i, and every $B_i \in \mathcal{T}$. So, we can conclude that C_n is open. Also, $x \in B_i$, therefore $x \in C_n$. So, actually, this $\mathcal{B}_x$ is a collection of open sets containing x. Further, if we are taking another open set $G \in \mathcal{T}$ and G is containing x. Then, as this is a countable basis at x, we can find some B_{n_0}, where $n_0 \in N$ such that $x \in B_{n_0} \subseteq G$. But from the construction of C_n, we can also find a $C_{n_0} \subseteq B_{n_0}$ and that is a subset of G, or $x \in C_{n_0} \subseteq G$. Thus, we conclude that this $\mathcal{B}_x$ is a countable basis at x.

Moving ahead, let us see this result regarding convergence of a sequence. If we are having a topological space $(X, \mathcal{T})$ and $A \subseteq X$. If there is a sequence of points of A converging to X, then $x \in cl(A)$. In order to deduce this result, let us recall a result regarding $cl(A)$, that is, $x \in cl(A)$ if and only if for all open sets G containing x, $G \cap A$ is non-empty. We will use this result. So, if we are taking a sequence (x_n), where this sequence is from A and what we are saying is that (x_n) is converging to x. As (x_n) is converging to x, for all open sets G containing x, there exists a natural number n_0 such that $x_n \in G$ for all $n \geq n_0$. From here, we conclude that $A \cap G$ is non-empty, or $x \in cl(A)$. If we are looking for the converse of this result, note that the converse holds when we assume the space as a first countable space; that is if $(X, \mathcal{T})$ is a first countable space then if $x \in cl(A)$, there exists a sequence of points of A converging to x. In order to prove this result, begin with a first countable space $(X, \mathcal{T})$. It is given that this space is first countable and if this space is first countable, for all $x \in X$, there exists a countable basis, say the basis is this $\mathcal{B}_x = \{B_n : n \in \mathbb{N}\}$ such that $B_{n+1} \subseteq B_n$, $n = 1, 2, ...$ Now, if we are taking $x \in cl(A)$, then we note that for all open sets G containing x, $G \cap A$ is non-empty, and obviously, this G is a non-empty open set. Note that we have considered this B_n as a member of topology because this $\mathcal{B}_x$ is a basis. Therefore, we can say that this $B_n \cap A$ is also non-empty for all n. Now, let us consider an element of this $B_n \cap A$. So, we can get a sequence (x_n) and the elements of the sequence are from A, that is this is a sequence in A. Now, if we want to show that this sequence converges to x, let us take any open set G and which is containing this x. Then, by the definition of this basis, there exists an element $B_{n_0} \in \mathcal{B}_x$ such that $x \in B_{n_0} \subseteq G$. From here, what can we conclude? We can conclude that $x_n \in G$, for all $n \geq n_0$ because of the property available with this basis as this B_{n_0} is a superset of B_n, for all $n \geq n_0$. Therefore, the sequence (x_n) converges to x. That's all about the proof of this theorem.

Now moving to the next concept, that is the concept of second countable spaces. A topological space having a countable basis is said to satisfy the second countability axiom, or that is known as a second countable space. Precisely, if we are having a topological space $(X, \mathcal{T})$ if we can construct a basis for it and this basis is something like $\{B_n : n \in \mathbb{N}\}$, because we require a countable basis, then this space turns out to be a second countable space.

Let us take some examples. The well-known example, that is the real line or set of real numbers with the Euclidean topology. We also say the Euclidean topology, the usual or standard topology. This space is second countable because we can construct a countable basis for it, and that is a well-known collection $\{(a, b) : a, b \in \mathbb{Q}\}$. Even if we are taking $\mathbb{R}^n$ with the Euclidean topology, that's also second countable because it has a countable basis given by the collection $\{(a_1, b_1) \times (a_2, b_2) \times ... \times (a_n, b_n) : a_i, b_i \in \mathbb{Q}, 1 \leq i \leq n\}$.

Moving ahead, let us take some of the examples which are not second countable. The real line with lower limit topology $(\mathbb{R}, \mathcal{T}_l)$, we have seen that this space is a first countable space, but interestingly, this is not a second countable space. In order to justify that this real line with lower limit topology is not second countable, assume that this is second countable and if this topological space is second countable there exists a countable basis. Let us denote that by $\mathcal{B}$. Now, if we are taking $x \in \mathbb{R}$, we know that $[x, x + 1) \in \mathcal{T}_l$ and also

$x \in [x, x+1)$. Therefore, by the definition of basis, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, x+1)$. Now, if we are taking any $y \in R$ such that y is not equal to x, so this is clear that this semi-open interval $[x, x+1)$ cannot be equal to $[y, y+1)$ because it will look like something if this is x and this is $x+1$, then the second will be something like if this is y then this is $y+1$. Also, we can justify that if there exists $B_y \in \mathcal{B}$ such that $y \in B_y \subseteq [y, y+1)$ then B_x cannot be equal to B_y. This is possible because as this B_x is a subset of $[x, x+1)$, if we are computing the infimum of this B_x, that will always be greater than or equal to the infimum of $[x, x+1)$, which is always equal to x. Also, as $x \in B_x$, infimum of B_x is always less than or equal to x. Therefore, the infimum of this B_x will always be x, and similarly, the infimum of this B_y is equal to y. So, from here it is clear that B_x cannot be equal to B_y. Now, if we are defining a function $f : \mathbb{R} \to \mathcal{B}$ such that $f(x) = B_x$, then what we have already justified here is that if x is not equal to y, B_x cannot be equal to B_y. Thus, function f is an injective function, but what about $\mathbb{R}$? $\mathbb{R}$ is uncountable, and this $\mathcal{B}$ is countable. So, we have constructed an injective function from an uncountable set to a countable set, which is not possible, that's why we reach to a contradiction. Therefore, $(\mathbb{R}, \mathcal{T}_l)$ is not second countable.

Let us take another example: an uncountable set with discrete topology is also not second countable, while we have already seen that this topological space is a first countable space. Why? Again, we justify by contradiction. For example, if we are taking this is $(X, \mathcal{T})$, and we are assuming that this is second countable. It means that it has a countable basis. Let us take that countable basis be this $\mathcal{B}$. Also, we know that for all $x \in X$, $\{x\} \in \mathcal{T}$. Therefore, by using the definition of basis, there exists some $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq \{x\}$. From here, we can conclude that this B_x is nothing but $\{x\}$. But note that X is an uncountable set. So, what about B? If we are considering this $\mathcal{B}$ as $\{B_x : x \in X\}$, this becomes uncountable. Therefore, this is a contradiction. Hence an uncountable set with discrete topology is not second countable.

Now, let us discuss the relationship between the first and second countable spaces. We have already seen that first countable spaces depend on the countable basis at each point, while the concept of second countable spaces is based on the countable basis for space. But we can deduce that every second countable space is a first countable space. In order to justify this result, let us take a second countable space. We are taking that space $(X, \mathcal{T})$. Because this is a second countable space, it has a countable basis. Let us denote that countable basis by $\mathcal{B}$. Now, if we are taking any $x \in X$, let us take a collection $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$. Now, what will or what property does this $\mathcal{B}_x$ have? $\mathcal{B}_x$ is a collection of open sets containing x. Now, if we are taking any open set G containing this x, by the definition of basis, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq G$. But note that such B we have already taken in $\mathcal{B}_x$. It means that there exists $B \in \mathcal{B}_x$ with the property that $x \in B \subseteq G$. Therefore, this $\mathcal{B}_x$ is basis at x. But at the same time, it is countable, too, because this $\mathcal{B}$ is countable. Thus, we have seen that corresponding to each and every element of $x \in X$ there exists a countable basis at x. Thus, every second countable space is a first countable space. But note that the converse is not necessarily true. Why? The answer is, we have already seen that if we are taking an uncountable set with discrete topology, we have already justified that this space is first countable but this space is not second countable. Therefore, the converse of this result is not true.

Moving ahead, let us see the behavior of subspaces of a second countable space, and our result is: a subspace of a second countable space is second countable. In order to justify this theorem, let us take a second countable space $(X, \mathcal{T})$. This space is second countable and because this is second countable, it will have a countable basis. Let us take that countable basis be denoted by this $\mathcal{B}$. Now, if we are taking a subset Y of X and we want to justify that this subspace $(Y, \mathcal{T}_Y)$ is also second countable, we have to construct a countable basis for it. Let us take this $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$. Note that because $\mathcal{B}$ is countable, this $\mathcal{B}_Y$ is countable too. Also, we have already seen that if this $\mathcal{B}$ is the basis for $(X, \mathcal{T})$, then this $\mathcal{B}_Y$ is the basis for $(Y, \mathcal{T}_Y)$. Therefore, $(Y, \mathcal{T}_Y)$ has a countable basis. Hence, the result, that is, a subspace of a second countable space is second countable.

These are the references.

That's all from this lecture. Thank you.