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Professor Name: S.P. Tiwari
Department Name: Mathematics & Computing
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Welcome to Lecture 24 on Essentials of Topology.

In this lecture, we will study the concept of Kuratowski closure and interior operators. Actually, this is another way to introduce topology by using some operators. Let us have a glimpse of what we are going to do here. The idea is if we are having a nonempty set X. Let us try to define a map $c : P(X) \to P(X)$. The question is whether we are taking an arbitrary map from $c : P(X) \to P(X)$ or we are putting some restrictions on it. The answer is: We will put some restrictions and what restrictions we are putting here that are inspired from the properties of closure of a set. Basically, what we have already seen is that if we are taking an empty set, the closure of the empty set will always be an empty set. We also know that every set is a subset of its closure. Even if we are taking $cl(A \cup B)$, that is given by $cl(A) \cup cl(B)$. One more property of closure we know, that is $cl(cl(A))$ is always equal to $cl(A)$, where A and B are subsets of X. Even there are some more features available with the closure, but these properties only will help us to introduce a topology on X by using this map.

Exactly, if we are having a nonempty set X, let us take a map $c : P(X) \to P(X)$. This map is called Kuratowski closure operator if

1. $c(\emptyset) = \emptyset$,
2. $A \subseteq c(A)$,
3. $c(A \cup B) = c(A) \cup c(B)$, and
4. $c(c(A)) = c(A)$,

for all $A, B \in P(X)$. From this concept of the Kuratowski closure operator, we can observe that if $A \subseteq B$, then $c(A) \subseteq c(B)$, and this is simple to deduce because, from this third one, that is $c(A \cup B) = c(A) \cup c(B)$, if $A \subseteq B$, what will happen? It becomes $c(B) = c(A) \cup c(B)$, and from here, we can conclude that $c(A) \subseteq c(B)$. This is a similar property satisfied by closure of a set. Let us see some of the examples of Kuratowski closure operators.

If we are taking a map $c : P(X) \to P(X)$, defined by $c(A) = A$, for all $A \in P(X)$. Note that this is precisely an identity map on $P(X)$. This is one of the examples of the Kuratowski closure operator because, from the definition itself, it is clear that $c(\emptyset) = \emptyset$ as $c(A) = A$. We can always think of $A \subseteq c(A)$. Now, if we are taking any two subsets, A and B of X, and if we are looking at $c(A \cup B)$, by definition, that will be nothing but $A \cup B$, and that is $c(A) \cup c(B)$. Finally, if we are thinking about $c(c(A))$, that will be equal to $c(A)$. So, the identity map on $P(X)$ is always a Kuratowski closure operator.

Let us take another example. For a nonempty set X and $A \subseteq X$, let

$$c(A) = \begin{cases} \emptyset, & \text{if } A = \emptyset \\ X, & \text{otherwise.} \end{cases}$$

Then this c is a Kuratowski closure operator. If we want to justify or we want to visualize it, it is clear that $c(\emptyset) = \emptyset$. Also, $A \subseteq c(A)$, this is for all A subsets of X, which holds from this definition. Now, if we want to justify that $c(A \cup B) = c(A) \cup c(B)$, this is also a simple one because if we are taking one set as an empty set and the other set as a nonempty set, then what will happen? This $c(A \cup B) = X$, and this $c(A) \cup c(B)$ will be $\emptyset \cup X$, that becomes X. But in the case when we are taking this A as nonempty set and B as also a nonempty set, then again this $c(A \cup B)$, that will be equal to X and what about this $c(A) \cup c(B)$, that will also be equal to X. Therefore, $c(A \cup B) = c(A) \cup c(B)$. Finally, if A is an empty set and if we want to compute $c(c(A))$, that will always be an empty set, but if A is a nonempty set, then if we want to compute $c(c(A))$, then the $c(c(A)) = X$, and it can also be written as $c(A)$. Thus, the map c defined here is a Kuratowski closure operator.

Let us take one more general example. For an infinite set X and $A \subseteq X$, let

$$c(A) = \begin{cases} A, & \text{if } A \text{ is finite} \\ X, & \text{otherwise.} \end{cases}$$

Then c is a Kuratowski closure operator. Let us see it. Because $c(A) = A$ if A is finite, therefore if we are taking A as an empty set, obviously, $c(\emptyset) = \emptyset$.

Now, if we are taking any subset A of X. From the definition, this is clear that $c(A)$ is always a superset of A, or we can say that A is a subset of $c(A)$. Now, if we want to justify the behavior of this c on the union of two sets, let us take two subsets, A and B of X. We want to compute, or we want to show that $c(A \cup B) = c(A) \cup c(B)$. There may be different cases. Let us take one by one. The first case is if A and B are finite sets. In this case, $c(A \cup B) = A \cup B$, and if we are computing $c(A) \cup c(B)$, that will be nothing but $A \cup B$. So, there is no problem in this case. Moving ahead, let us take one set as a finite set and another as an infinite set. Why not let us take A as a finite set, and we are taking this B as an infinite set? What will happen if we want to compute this $c(A \cup B)$? Note that A is finite and B is infinite. So this $A \cup B$ becomes an infinite set, and therefore, $c(A \cup B) = X$. Now, if we want to compute $c(A) \cup c(B)$, then by definition, $c(A) = A$, and $c(B) = X$, and that's why $c(A) \cup c(B) = X$. So, again it holds. Finally, let us take a third case when A and B, both are infinite sets. What will happen in this case if we compute $c(A \cup B)$. By the definition because both are infinite sets, this will always be equal to X, and if we compute $c(A) \cup c(B)$, that will be $X \cup X = X$. Coming to the last point that is if we want to compute $c(c(A))$. If A is finite, what will happen? This $c(c(A))$, that will be equal to $c(A)$ because A is finite. If we are taking A as an infinite set then what will happen? $c(c(A))$, that will be nothing but $c(X) = X$, and that can be taken as $c(A)$. Therefore, this c is a Kuratowski closure operator. Similar to this example, instead of taking an infinite set, let us take an uncountable set and define the map $c : P(X) \to P(X)$ as under:

$$c(A) = \begin{cases} A, & \text{if } A \text{ is countable} \\ X, & \text{otherwise.} \end{cases}$$

Similar to the previous example, we can justify that c is a Kuratowski closure operator.

Now, if we are having a Kuratowski closure operator, the question is how we can talk about a topology on X. The answer is this particular theorem. This theorem says that if c is a Kuratowski closure operator on this X, then there always exists a topology on X such that $c(A) = cl(A)$, for all $A \in P(X)$. One of the interesting things, and that is a fact available in the statement of the theorem, is because $c(A) = cl(A)$, and we know that $cl(A)$ is always a closed set, it means that $c(A)$ is going to be a closed set. From here, we can construct a topology, so existence is inspired by this particular fact. Now, let us begin with a family $f = \{A \subseteq X : c(A) = A\}$, and by using this here, let us construct a family $\mathcal{T}$, that is a collection $\{G \subseteq X : G^c \in f\}$. We can show that this $\mathcal{T}$ is a topology on X, and finally, we will justify that with respect to this topology, $c(A) = cl(A)$.

In order to justify that this $\mathcal{T}$ is a topology on X, let us first justify that $\emptyset \in \mathcal{T}$. This is possible because $\emptyset^c = X \in f$. The question is how? The answer is, if we want to justify that $X \in f$, we want to justify that $c(X) = X$, and this is equivalent to justifying that $X \subseteq c(X)$ and $c(X) \subseteq X$. $X \subseteq c(X)$ is always true because we know that if c is a Kuratowski closure operator, for all subsets A of X, $A \subseteq c(A)$, and $c(X) \subseteq X$, because $c : P(X) \to P(X)$, which is sending each subset A of X to a subset of X. Therefore $c(X) \subseteq X$. So, we justify that $\emptyset \in \mathcal{T}$. Also, $X \in \mathcal{T}$ because we know that the $X^c = \emptyset \in f$, as we have already seen that $c(\emptyset) = \emptyset$.

Moving ahead, let us try to justify that if we are taking a finite number of subsets $G_1, G_2, ..., G_n$ of X, which are in $\mathcal{T}$, justify that $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$. What are we going to do here? We are going to justify for two sets, and after that, by induction, we can justify for all n-sets. So, if we are taking $G_1, G_2 \in \mathcal{T}$, and we want to show that $G_1 \cap G_2 \in \mathcal{T}$. For which, we have to justify that $(G_1 \cap G_2)^c \in f$, or $G_1^c \cup G_2^c \in f$. But if we want to show that this is in f, we have to show that the image under this c, that is, $c(G_1^c \cup G_2^c) = G_1^c \cup G_2^c$. But the question is, is it possible? The answer is yes because $G_1, G_2 \in \mathcal{T}$. Therefore, $G_1^c, G_2^c \in f$, that is, $c(G_1^c) = G_1^c$, and $c(G_2^c) = G_2^c$. By the property of the Kuratowski closure operator, $c(G_1^c \cup G_2^c)$ can be written as $c(G_1^c) \cup c(G_2^c)$, and because these are individually equal to the set itself, so this will be $G_1^c \cup G_2^c$. That's all about this justification.

Moving ahead, let us take an indexed family of subsets $\{G_i : i \in I\}$ from $\mathcal{T}$, that is, $G_i \in \mathcal{T}$. Then, our motive is to justify that the $\cup\{G_i : i \in I\} \in \mathcal{T}$. For which, we have to justify that $(\cup\{G_i : i \in I\})^c \in f$, or by De Morgan's law, $\cap\{G_i^c : i \in I\} \in f$. In other words, we have to justify that $c(\cap\{G_i^c : i \in I\}) = \cap\{G_i^c : i \in I\}$. This will be equivalent to justifying two things. The first is to justify that $\cap\{G_i^c : i \in I\} \subseteq c(\cap\{G_i^c : i \in I\})$, and the second step will be to justify that $c(\cap\{G_i^c : i \in I\}) \subseteq \cap\{G_i^c : i \in I\}$. From the definition of c itself, the first is trivial because we know that $A \subseteq c(A)$. Coming to the second one. From the set theory, we already know that $\cap\{G_i^c : i \in I\} \subseteq G_i^c$, for all $i \in I$. Now, as we have already seen that if $A \subseteq B \Rightarrow c(A) \subseteq c(B)$. Therefore, $c(\cap\{G_i^c : i \in I\}) \subseteq c(G_i^c) = G_i^c$. Thus $c(\cap\{G_i^c : i \in I\}) \subseteq \cap\{G_i^c : i \in I\}$. This is the justification of this part, and hence the $\mathcal{T}$ which we have taken as a collection of subsets G of X such that $G^c \in f$ is a topology on X.

Note that f we have taken as a collection of subsets A of X such that $c(A) = A$. Finally, we have to show that $c(A) = cl(A)$, for all $A \in P(X)$. Note that what we have shown is that this $\mathcal{T}$, which is a collection of all those subsets G of X such that $G^c \in f$, is a topology, and also, this f was nothing but a collection of all those subsets A of X such that $c(A) = A$. From the definition of this $\mathcal{T}$, it is clear that this f is a family of closed subsets of X because it contains the complements of open subsets. Now, with respect to this topology, one thing is clear: $cl(A)$ is a closed set, and if $cl(A)$ is a closed set, what will happen? This $cl(A)$ is an element of f, and if this is an element of f what will happen? $c(cl(A)) = cl(A)$. Also, we know that $cl(A)$ contains A, by the definition of c, we have seen that $A \subseteq c(A)$. As $c(c(A)) = c(A)$, it means that $c(A) \in f$. But note that the elements of f are closed with respect to the topology, therefore $c(A)$ is also a closed set. Now, we have two closed sets, $cl(A)$ as well as $c(A)$, both containing A. But by the definition of $cl(A)$, we know that this $cl(A)$ is the smallest closed set, and therefore, we can conclude that $cl(A) \subseteq c(A)$. Finally, we want to justify one more thing here: $c(A) \subseteq cl(A)$. We know that $A \subseteq cl(A)$. By the definition of c, $c(A) \subseteq c(cl(A))$, and what we have seen is that $c(cl(A)) = cl(A)$; therefore, $c(A) \subseteq cl(A)$. Clubbing these two, that is, $c(A) \subseteq cl(A)$ and $cl(A), \subseteq c(A)$, we conclude that $c(A) = cl(A)$.

Moving to the next, let us take some of the examples. Beginning with the first one which we have seen that if we are having a nonempty set X, $c : P(X) \to P(X)$ such that $c(A) = A$. We have already seen that c is a Kuratowski closure operator, where $A \subseteq X$. Now, in view of the previous theorem, f is a collection of all those subsets A of X such that $c(A) = A$, and by using this definition, this f is nothing but$P(X)$. If we are defining the topology $\mathcal{T}$, that is nothing but $P(X)$. So, what we can say that this $\mathcal{T}$ is nothing but the discrete topology on X. Meaning is if c is an identity map on $P(X)$, then the topology induced by c on X is a discrete topology. Let us move to another example which we have already seen that if we are taking a nonempty set X with a map $c : P(X) \to P(X)$ and the map was defined like $c(A) = \emptyset$, if $A = \emptyset$

and $c(A) = X$, otherwise. If we are writing f and $\mathcal{T}$ in this case, what will be our f? f will be nothing but a collection of subsets of X such that $c(A) = A$, and that is nothing but the collection of empty sets as well as X. So, what about $\mathcal{T}$ in this case? See, the topology is indiscrete. It means that by using this particular Kuratowski closure operator, we can induce indiscrete topology on X.

Let us take one more example. Begin with an infinite set. We have seen that c is a Kuratowski closure operator, which was defined as $c(A) = A$ if A is finite and $c(A) = X$, otherwise. Now, what will be f in this particular case? f is nothing but a collection of all subsets A of X such that A is finite because if A is finite, $c(A) = A$. Also, $c(X) = X$, that's why X will also come in f. Now, if we are looking for, what the topology is? This topology is a collection of all those subsets of X such that either G is an empty set or G^c is finite and recall the concept that is nothing but the co-finite topology on X. Similarly, one more example we have seen, and one can check that the topology induced by that Kuratowski closure operator is nothing but the co-countable topology on X.

Similar to the concept of closure operator, we have its dual concept; in another sense, by using the interior of a set, we can also define a topology. What the concept is? If we are having a nonempty set X, let us define a map $i : P(X) \to P(X)$. We say that this i is Kuratowski interior operator if it satisfies, $i(X) = X$, $i(A) \subseteq A$, $i(A \cap B) = i(A) \cap i(B)$, and $i(i(A)) = i(A)$, for all $A, B \subseteq X$. It can be seen that these are the properties of the interior, that is if we are having a topological space $(X, \mathcal{T})$, $A \subseteq X$, then $int(A)$ satisfies all these four. For example, the simplest one, let us take $i : P(X) \to P(X)$, $i(A) = A$. Then we can justify that this i is a Kuratowski interior operator. Also, similar to the concept of the Kuratowski closure operator, we can talk about the topology induced by this Kuratowski interior operator. The theorem is, if i is the Kuratowski interior operator on a nonempty set X then there exists a topology on X such that $i(A) = int(A)$, for all $A \in P(X)$. Note that how we are taking this $i(A)$, we are taking $i(A)$ as the interior of A, and we know that the interior of A is an open set. So, whenever we want to justify the existence of topology, the answer can be given in a simple way, that if we are considering $\mathcal{T}$ as a collection $\{G \subseteq X : i(G) = G\}$, then we can justify that $\mathcal{T}$ is a topology on X. Even we can also prove that $i(A) = int(A)$. It can be observed that when we are taking the map i as $i(A) = A$, then it can induce a topology on X, and that topology is nothing but the discrete topology. Similarly, we can discuss other examples.

These are the references.

That's all from this lecture. Thank you.