

Course Name: Essentials of Topology
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Welcome to Lecture 23 on Essentials of Topology.

In this lecture, we will study the concept of quotient topology, which is another way for the creation of a new topology. Let us have a glimpse on the methods for creation of new topologies, which we have already seen. We have seen that if we are having a topological space (X, \mathcal{T}) and a subset Y of X , then we can put a topology on Y , that is \mathcal{T}_Y , and we call it the relative topology or subspace topology. What another way we have seen that is the concept of product topology. It means that if we are having two topological spaces, then we can talk about topology on the Cartesian product of X_1 and X_2 , and we have seen the concept of product topology.

What we are going to do here, that is if we are having a topological space (X, \mathcal{T}) , and let us have a map $q : X \rightarrow A$. This map is a surjective map, and whenever we are talking about A , this is a non-empty set, even if this is not necessarily a subset of X . The question is, by using this topology \mathcal{T} , can we put a topology on A ? The answer is yes. We can do it by using this particular map, that is the surjective map q . How to do it? Let us see. If we are having a topological space (X, \mathcal{T}) and a non-empty set A . If we are taking this $q : X \rightarrow A$, this is a surjective map, then

$$\mathcal{T}_q = \{G \subseteq A : q^{-1}(G) \in \mathcal{T}\}$$

is a topology on A . Let us see how this \mathcal{T}_q is a topology.

The first one, this empty set will always be a member of this \mathcal{T}_q because the empty set is a subset of A and also $q^{-1}(\emptyset) = \emptyset \in \mathcal{T}$. Also, $A \in \mathcal{T}_q$ because A is a subset of itself, and if we are computing $q^{-1}(A)$, that will be nothing but X , and $X \in \mathcal{T}$. Moving ahead, let us take a family or a finite family of subsets of A , say, $G_1, G_2, \dots, G_n \in \mathcal{T}_q$. Our motive is to justify that the intersection of this finite family, that is, $G_1 \cap G_2 \cap \dots \cap G_n \in \mathcal{T}_q$. Now,

$q^{-1}(G_1 \cap G_2 \cap \dots \cap G_n) = q^{-1}(G_1) \cap q^{-1}(G_2) \cap \dots \cap q^{-1}(G_n)$. When we are taking $G_1, G_2, \dots, G_n \in \mathcal{T}_q$, it means that $q^{-1}(G_i) \in \mathcal{T}$, where $1 \leq i \leq n$. Because this $q^{-1}(G_i) \in \mathcal{T}$, we can say that this $q^{-1}(G_1) \cap q^{-1}(G_2) \cap \dots \cap q^{-1}(G_n) \in \mathcal{T}$, or $G_1 \cap G_2 \cap \dots \cap G_n \in \mathcal{T}_q$.

Moving ahead, let us take an indexed family of sets, that is $\{G_i : i \in I\}$, where $G_i \in \mathcal{T}_q$. Our motive is to justify that $\cup\{G_i : i \in I\} \in \mathcal{T}_q$. Now, when we are saying that $G_i \in \mathcal{T}_q$, it means that $q^{-1}(G_i) \in \mathcal{T}$, and this is for all $i \in I$. If we are computing an inverse image of $\cup\{G_i : i \in I\}$ under q , note that this can be written as $\cup\{q^{-1}(G_i) : i \in I\}$. But note that this $q^{-1}(G_i) \in \mathcal{T}$, for all $i \in I$. Therefore, $\cup\{q^{-1}(G_i) : i \in I\} \in \mathcal{T}$, and therefore, we conclude that $\cup\{G_i : i \in I\} \in \mathcal{T}_q$, or this \mathcal{T}_q is a topology on A .

If this \mathcal{T}_q is a topology on A , what have we seen? We begin with a topological space (X, \mathcal{T}) , we take a map $q : X \rightarrow A$, and we get a topology on A , and we denoted this topology by \mathcal{T}_q . So, what we call, we call this \mathcal{T}_q , the quotient topology induced by map q , and this q itself is known as the quotient map, and the topological space (A, \mathcal{T}_q) is known as the quotient space.

Let us take some of the examples. Begin with the set of real numbers with Euclidean topology. Let us take a set $A = \{a, b, c\}$. Define a map $q : \mathbb{R} \rightarrow A$ such that $q(x) = a$ if $x < 0$, $q(x) = b$ if $x = 0$, and $q(x) = c$ if $x > 0$. If we want to see that, what will this \mathcal{T}_q be? Note that A is consisting of three elements, therefore there will be eight subsets. Obviously, $q^{-1}(\emptyset) = \emptyset$, and this $q^{-1}(A) = \mathbb{R}$. But if we are looking for what about $q^{-1}(\{a\})$? Note that $q^{-1}(\{a\}) = (-\infty, 0)$, and this is a member of Euclidean topology on \mathbb{R} . But if we are looking for $q^{-1}(\{b\})$, that will be nothing, but $\{0\}$, which is not a member of the Euclidean topology. Even if we are computing $q^{-1}(\{c\})$, that is nothing but $(0, \infty)$, and that is a member of the Euclidean topology. Moving ahead, if we are taking an inverse image of two points set, for example, if we are computing $q^{-1}(\{a, b\})$, this will be nothing but $(-\infty, 0]$, which is not a member of the Euclidean topology. Similarly, what about $q^{-1}(\{b, c\})$? That will be nothing but $[0, \infty)$ which is not a member of Euclidean topology. But if we are taking $q^{-1}(\{a, c\})$ that will be nothing but $(-\infty, 0) \cup (0, \infty)$. So, from here, we can conclude that this quotient topology on A is given by $\mathcal{T}_q = \{\emptyset, A, \{a\}, \{c\}, \{a, c\}\}$.

If we are making a simple change, why not let us take this \mathbb{R} equipped with the lower limit topology. If we are looking for the quotient topology in this case, that will get changed and that can be given by $\mathcal{T}_q = \{\emptyset, A, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Meaning is, the topology, that is, the quotient topology on A will depend on the nature of the topology with which you are beginning with, and it can be seen that the topology will also depend on the map from where we are starting.

Let us visualize the quotient map here. What exactly are we doing? We are taking the set of real numbers. Let us take this is 0; how have we defined the map? Actually, this map divides the real line into three parts and maps to exactly three points, this is the point a , this is point b , and this is a point c . How is this mapping? This portion of the real line is mapping to a , we are mapping 0 to b , and we are mapping this portion to c . Exactly what are we doing? Actually, we are making a partition of the real line, and obviously, their union is equal to \mathbb{R} , and they are disjoint in nature.

So, what we have seen till now? We have seen the concept of quotient topology as well as an example. But what is the role of the surjective map which we are using, that we have not identified, that is, even when we have shown that \mathcal{T}_q is a topology or even in the previous example, we have not identified the power of the quotient map, which we have considered, and we have taken that map as a surjective map. Let us see the importance of surjectivity of q , which we are taking, that is, we have begun with a surjective map $q : X \rightarrow A$. So, how this is playing the role? Let us take this as the given set X , this is the given set A , and what are we taking, we are taking a map $q : X \rightarrow A$, and that we have considered a surjective map. Because of the surjectivity of q , this guaranteed that for all $a \in A$, there is some $x \in X$ such that $q(x) = a$. So, if we are using this concept, what exactly we are getting, that is from this definition if we are taking an element $a \in A$, when we are computing its inverse, we will get some of the part of this X and the important thing is that this $q^{-1}(\{a\})$ will always be non-empty. Similarly, we are taking another element $b \in A$, again what can we get? We can get some other part of this X . So, this one is precisely $q^{-1}(\{b\})$, and because of the surjectivity, this is guaranteed that this will always be non-empty. So, by using this idea, what can we do? We can make a partition of X , that is this X can be written as a union of $q^{-1}(\{a\}), a \in A$.

What are we making? We are making a partition of X , and for the partition, what sets are we using? Actually, we are using non-empty disjoint sets. Whenever we are talking about such partitions in terms of non-empty disjoint sets of any set, we know the well-known concept, that is, the concept of equivalence relations and equivalence classes. So, a natural question arises about how equivalence classes and equivalence relations play a role when we are discussing the concept of quotient topology. It's a simple one, but before going to discuss in detail the quotient topology induced by equivalence relations, let us recall the concept of equivalence relation.

If we are having a non-empty set X , we know that a relation R is a subset of $X \times X$. Also, we say that this relation is equivalence if it satisfies some properties, that is $(x, x) \in R$, second one if $(x, y) \in R$ this implies that $(y, x) \in R$, and also if $(x, y) \in R$ and $(y, z) \in R$ it implies that $(x, z) \in R$, this is for all $x, y, z \in X$. We are familiar with these notions. The first one is known as reflexivity, the second is symmetry of R , and the third is transitivity of R . Now, we are having an equivalence relation, that is if R is an equivalence relation, then we can talk about the equivalence class of an element $x \in X$, that is denoted by $[x]$, this is given as $\{y \in X : (x, y) \in R\}$. The important thing which we all know about equivalence classes is that two equivalence classes are either equal or they are always disjoint. Also, we know that this set X can be written as a union of $[x], x \in X$, or equivalence classes make a partition of the set X . Now, let us take a set, called the factor set denoted by X/R , this as a collection $\{[x] : x \in X\}$, that is this is our factor set, then a natural map is here, that map is $q : X \rightarrow X/R$, which is defined as $q(x) = [x]$. Note that this map is a surjective map. Now, the interesting thing is here that we always have a surjective map from a set to its factor set. Now, if we are having this as a surjective map what can we do? We can talk about the quotient topology, that is if X is having a topology, that is \mathcal{T} is a topology on X . Then we can discuss the quotient topology on this factor set induced by this map, and this map is a surjective map defined as $q(x) = [x]$.

Now, let us take an example based on this concept. Begin with the set of integers, which is a subset of the set of real numbers. Let us take Euclidean topology on \mathbb{R} , and \mathbb{Z} is having the relative topology with respect to this Euclidean topology. We know that the topology will always be a discrete topology. Let us define a relation R on \mathbb{Z} . This relation is defined as: if we are

taking two integers x and y , we say that $(x, y) \in R$ if and only if $y - x$ is an even integer. From the definition, this is simple to observe that this relation is always an equivalence relation, that is R is an equivalence relation. If we are looking for the equivalence classes, so what will happen? $[0] = [2] = [4] = \dots$ Similarly, $[1] = [3] = [5] = \dots$ So, what will happen if we want to visualize that what $[0]$ is? This is $\{\dots, -4, -2, 0, 2, 4, \dots\}$. Similarly, what about $[1]$?, that is nothing but $\{\dots, -3, -1, 1, 3, \dots\}$. So, its factor set, that is $\mathbb{Z}/R = \{[0], [1]\}$ with the natural surjective map that is sending integers to its equivalence classes. Now, we can talk about the quotient topology on this factor set \mathbb{Z}/R . Obviously, if we are looking for topology here, that is $\{\emptyset, \mathbb{Z}/R, [0], [1]\}$. It is clear that this topology is also discrete. So, what we have seen here that if we are defining an equivalence relation on the set of integers which is defined in this particular fashion, we can find a quotient topology on the factor set induced by this relation. We begin with the discrete topology on \mathbb{Z} , and we have seen that the topology which we are getting on the factor set is also a discrete topology.

Moving ahead, let us take a more general example. This example is on the closed interval $[0, 1]$. Let us take a set X as a closed interval $[0, 1]$. This is a subset of the set of real numbers. Again, we are taking the Euclidean topology on the set of reals. So, we can talk about the relative topology on $X = [0, 1]$. Now, let us define an equivalence relation on this X in this way that if we are having a relation R , the equivalence classes are, one is $\{0, 1\}$, that is, end points of this closed interval, and rest we are taking the singleton sets $\{x\}$, for all $x \in X$, with $0 < x < 1$. In other way, we are identified 0 and 1 as a same point and after that rest of the points are different one. What exactly going on here, that is if this is the closed interval $[0, 1]$, we are taking these points in one equivalence class, these are our end points, and the rest of the points, we are sending to corresponding singleton sets. The important thing is, actually, we are taking endpoints in the same equivalence class, that is, the equivalence class of 0 and the equivalence class of 1 are the same, that is given by this set. So, what can we do that, or how can we visualize this concept? Thus, this concept can also be visualized in this particular fashion. Let us take a unit circle and when we are talking about the endpoints, this is corresponding to this equivalence class. The interval is already given to us, this is the closed interval $[0, 1]$. Note that the topology which we are taking on this closed interval $[0, 1]$, that is relative topology with respect to the Euclidean

topology. So, the open sets will be here, either of this form, that is inside this closed interval $[0, 1]$, or that can also be of this form, that is $[0, a)$, even that may look like of form $(b, 1]$, where $0 < a, b < 1$. If you want to visualize that what are the members of the quotient topology on this unit circle, so we can observe that if we are taking an arc here and we want to find out what will be its inverse image, the inverse image will look like in this way, that is one open interval inside this closed interval so these arcs will always be an open subset of this circle. But if we are taking an arc which is containing this particular point, that is corresponding to equivalence class of 0 as well as 1, then what will be the inverse image of this arc, that will look like in the form $[0, a) \cup (b, 1]$.

We have already seen that these are open subsets of closed interval $[0, 1]$, and therefore this will also be open. Now, what we can conclude about this quotient topology, just recall this quotient topology will nothing but that will be relative topology with respect to the Euclidean topology on \mathbb{R}^2 . So, what exactly have we done here, this is an interesting thing that if we are taking some string or thread something like this form and if we want to glue the end-points of the string, this will look like in this particular way, and we are having a map from here, that is a quotient map in a natural way that may become a surjective map and by using this surjective map we can put a topology on this structure.

These are the references.

That's all from this lecture. Thank you.