Course Name: Essentials of Topology
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Welcome to Lecture 22 on Essentials of Topology.

In this lecture, we will study the concepts of product and box topologies. Specifically, we will study the concept of topology on arbitrary products of sets. Let us recall the concepts we studied in the previous lecture. Begin with the concept of product topology. We have seen that there were two ways to characterize the product topology. One was by using the concept of basis and the second was by using the concept of sub-basis.

Begin with the characterization of product topology by using the concept of basis. What we have seen is that if we are having two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$, then $\mathcal{B} = \{G_1 \times G_2 : G_1 \in \mathcal{T}_1, G_2 \in \mathcal{T}_2\}$ is a basis for a topology (called, product topology) on $X_1 \times X_2$. We have also seen that we can reduce the size of this basis and that characterization was given in this particular fashion. That is, if we are having two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ with their basis as $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, then $\mathcal{B} = \{B_1 \times B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$ is a basis for product topology on $X_1 \times X_2$. Moving to the next, let us see the sub-basis for product topology on $X_1 \times X_2$. What we have seen is that if we are having two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ and the collection, $\mathcal{B}' = \{\pi_1^{-1}(G_1) : G_1 \in \mathcal{T}_1\} \cup \{\pi_2^{-1}(G_2) : G_2 \in \mathcal{T}_2\}$. Then $\mathcal{B}'$ is a subbasis for the product topology on $X_1 \times X_2$.

So, in the previous lecture what have we seen? Actually, we have tried to construct product topology on $X_1 \times X_2$, when the topology $\mathcal{T}_1$ was given on X_1 and $\mathcal{T}_2$ was gven on X_2. Note that this Cartesian product can be generalized for n-topological spaces. Meaning is to say that if we are having topological spaces $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2), ..., (X_n, \mathcal{T}_n)$, then this collection, that is $\mathcal{B} = \{G_1 \times G_2 \times ... \times G_n : G_i \in \mathcal{T}_i, i = 1, 2, ..., n\}$ is a basis for product topology on $\prod_{i=1}^{n} X_i = X_1 \times X_2 \times ... \times X_n$.

Moving ahead, similar to the concept of product topology on two sets, and if bases were given for only two topologies, the same concept can be generalized for n-topological spaces, meaning is to say that if we are having $\mathcal{B}_1$, $\mathcal{B}_2$,..., $\mathcal{B}_n$ as bases for the topological spaces $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$,..., $(X_n, \mathcal{T}_n)$, respectively. Then $\mathcal{B} = \{B_1 \times B_2 \times ... \times B_n : B_i \in \mathcal{B}_i, i = 1, 2, ..., n\}$ is a basis for the product topology on $\prod_{i=1}^{n} X_i$. Again, we can give the sub-basis characterization for product topology on this product of n-sets, that is if we are having n-topological spaces $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$,..., $(X_n, \mathcal{T}_n)$ and

$$\mathcal{B}' = \bigcup_{i=1}^{n} \{\pi_i^{-1}(G_i) : G_i \in \mathcal{T}_i\}.$$

Then $\mathcal{B}'$ is a subbasis for the product topology on $\prod_{i=1}^{n} X_i$.

Moving ahead, we have already seen the Euclidean topology on $\mathbb{R}^2$ by using the concept of basis as well as by using the concept of metric. So, it can be seen that the Euclidean topology on $\mathbb{R}^2$ is the same as the product topology on the set, $\mathbb{R} \times \mathbb{R}$, that is $\mathbb{R}^2$ because what we have seen that a collection of open rectangles forms a basis for the Euclidean topology on $\mathbb{R}^2$. When we are using the concept of product topology on $\mathbb{R}^2$, again the collection of open rectangles forms a basis for product topology on $\mathbb{R}^2$. This concept can be generalized for n number of sets for $n \geq 2$. We can conclude that the Euclidean topology on $\mathbb{R}^n$ is same as the product topology on the Cartesian product of these n-sets, that is given by $\mathbb{R}^n$.

Moving ahead, let us try to visualize the product of sets in a different way. Meaning is to say that if we are having n-sets, then the Cartesian product of $X_1, X_2, ..., X_n$, this was given as a collection of n-tuples $(x_1, x_2, ..., x_n)$, where $x_1 \in X_1$, $x_2 \in X_2$,..., $x_n \in X_n$. Note that what we are writing here, this is x_1, we say that this is corresponding to 1, this is x_2, this is corresponding to 2, and in the same way this is x_n corresponding to n. Now, this element $(x_1, x_2, ..., x_n)$ of $X_1 \times X_2 \times ... \times X_n$ can be visualized as a map $f : \{1, 2, ..., n\} \to \bigcup_{i=1}^{n} X_i$ such that this $f(i) \in X_i$, and by notation, this $f(i)$ is denoted as x_i.

So, what we can say is that the Cartesian product of these n-sets $X_1, X_2, ..., X_n$, this can be given as a collection of maps $f : \{1, 2, ..., n\} \to \bigcup_{i=1}^{n} X_i$ such that this $f(i) \in X_i$, for all $1 \leq i \leq n$. This Cartesian product can be denoted by $\prod_{i=1}^{n} X_i$. Now, what will happen if we are replacing this set by an arbitrary set, that is, an arbitrary index set, this is our motive, and if we are replacing this set by an arbitrary set I, we can define the product of an indexed family of sets. Let $\{X_i : i \in I\}$ be an indexed family of sets. Then

$$\prod_{i \in I} X_i = \{f : I \to \bigcup_{i \in I} X_i | f(i) \in X_i, \forall i \in I\}.$$

One point to be noted here is that whenever we are taking this indexed family of sets, we are assuming that this X_i is a non-empty set, for all $i \in I$. If X_i's are non-empty, their product will always be a non-empty set, and this is possible because of the axiom of choice.

Having the idea of an arbitrary product of sets in mind, let us see the concept of topology on this arbitrary product. What will we see? We will discuss two concepts here. One concept will be the concept of a topology on this product of sets by using sub-basis and another we will see by using the concept of basis. Begin with the concept of sub-basis, let $\{(X_i, \mathcal{T}_i) : i \in I\}$ be an indexed family of topological spaces. For $i \in I$, let $\pi_i : \prod_{i \in I} X_i \to X_i$ be a map such that $\pi_i(f) = f(i), f \in \prod_{i \in I} X_i$. One thing that will be interesting here is that if we are taking a G_i, which is a subset of X_i, and we are taking this G_i from the topology on X_i, the question is, what about the inverse of this G_i under this projection map π_i. By definition, this will be a collection of all maps in this product of sets, that is, $f \in \prod_{i \in I} X_i$ such that $\pi_i(f) \in G_i$, or it can be written as a collection of all $f \in \prod_{i \in I} X_i$ such that $f(i) \in G_i$, or it can be written as $\prod_{i \in I} A_i$, where this $A_i = G_i$, and $A_j = X_j$, for i not equal to j. Interestingly, if we are taking this G_i as X_i itself, what about this $\pi_i^{-1}(X_i)$? Obviously, this is nothing but $\prod_{i \in I} X_i$. Now, let us take a collection of such sets. These will provide the concept of a sub-basis for a topology on this product of sets. What exactly I want to say is that if we are taking $\mathcal{B}' = \bigcup_{i \in I} \{\pi_i^{-1}(G_i) : G_i \in \mathcal{T}_i\}$, this is obvious that $\mathcal{T}_i$ is a topology on X_i. Then, it forms a sub-basis for a topology. One thing is clear from here, if we are taking this X as $\prod_{i \in I} X_i$, we have already seen that $\pi_i^{-1}(X_i)$ is equal to X itself. Therefore, this collection will always be a sub-basis for a topology. Now, if this is a sub-basis, we know that beginning with a sub-basis, that is, $\mathcal{B}'$, we can move to a basis, and from basis, one can move to a topology.

Let us denote this product topology by $\mathcal{T}_p$. If we want to visualize that how an element of $\mathcal{B}$ can look like, then if we are taking any $B \in \mathcal{B}$, so this B can be written as $\pi_{i_1}^{-1}(G_{i_1}) \cap \pi_{i_2}^{-1}(G_{i_2}) \cap ... \cap \pi_{i_n}^{-1}(G_{i_n})$, that is a finite intersection of members of $\mathcal{B}'$. Then, this will be equal to the product of some sets A_i, $i \in I$ and how this A_i will look like? Actually, this A_i will be something like G_i, if $i \in \{i_1, i_2, ..., i_n\}$, and this A_i will be equal to X_i, otherwise. So, what we have seen is that in the case of the basis for product topology, which is obtained by using this $\mathcal{B}'$ as a sub-basis, an element of this basis can be expressed in the form of the product of sets, where these sets will look like in this form.

Moving ahead, let us see the topology on this Cartesian product by using the concept of basis. If we have an indexed family of topological spaces $\{(X_i, \mathcal{T}_i) : i \in I\}$, then this collection, that is,

$$\mathcal{B} = \{\prod_{i \in I} G_i : G_i \in \mathcal{T}_i\},$$

this is a basis for some topology on $\prod_{i \in I} X_i$. We call this box topology. It is going to be interesting here. We have already seen that if we are taking two topological spaces, even if we are taking n-topological spaces, then such collections provide the same topology on the Cartesian product of sets. But we are using a different name as both the topologies are going to be different. We will discuss this in detail after completing or after seeing this concept. So, the question is whether this $\mathcal{B}$ is a basis for some topology on the product of the sets. The answer is yes because if we are taking X as $\prod_{i \in I} X_i$. Then note that this X can be expressed as a union of members of this $\mathcal{B}$ because X itself is a member of this $\mathcal{B}$. Also, if we are taking two members from this $\mathcal{B}$, let us take $B_1, B_2 \in \mathcal{B}$. Then how this B_1 and B_2 will look like? Let us take this: B_1 is something like $\prod_{i \in I} G_i$, and B_2 is $\prod_{i \in I} H_i$. If we are taking any f in this $B_1 \cap B_2$, that is precisely $\prod_{i \in I} G_i \cap \prod_{i \in I} H_i$. This is nothing but this is $\prod_{i \in I}(G_i \cap H_i)$. Note that if we are taking this as B_3, this B_3 will also belong to $\mathcal{B}$ as $G_i, H_i \in \mathcal{T}_i$, this $G_i \cap H_i \in \mathcal{T}_i$. Thus, there exists $B_3 \in \mathcal{B}$ with the feature that $f \in B_3 \subseteq B_1 \cap B_2$. Therefore, this collection is a basis for some topology on the product of these sets.

Let us summarize what we have seen that if we are having an indexed family $\{(X_i, \mathcal{T}_i) : i \in I\}$ of topological spaces, then the box topology, let us denote this by $\mathcal{T}_b$ on $\prod_{i \in I} X_i$ has as basis elements, the sets of this form, that is $\prod_{i \in I} G_i, G_i \in \mathcal{T}_i$, for all $i \in I$. If we are coming to another topology, what we have seen is that the topology was the product topology on this set. Let us denote this topology by $\mathcal{T}_p$. Note that this topology has as a basis all sets of the form $\prod_{i \in I} G_i, G_i \in \mathcal{T}_i$, for all $i \in I$ and G_i equals X_i except for finitely many values of i. From these two, it is clear that in case I is finite, these topologies, that is, box topology and product topology, will be the same because their bases will be the same. But what about if this index set I is not finite. For example, why not let us take this set I as the set of natural numbers? One can see that this product topology is strictly coarser than this box topology. The question is, how is it possible? Let us see it. If we are taking a member G of this product topology, then how can this G be expressed? G can be expressed as a union of members of basis, and what we have seen that for product topology, the members of basis will look like in this form, that is $\prod_{i \in I} G_i, G_i \in \mathcal{T}_i$, for all $i \in I$ and G_i equals X_i except for finitely many values of i. But note that such products are also the members of the basis for box topology. Therefore, this G will always be in box topology; that is, this product topology will be coarser than box topology. But if we are taking an element of this box topology, in this particular form, that is G which is given as $\prod_{i \in I} G_i, G_i \in \mathcal{T}_i$. Note that every G_i is a proper subset of X_i. Then what will happen? From the construction, it is clear that this G is a member of the basis for box topology, and as the basis is always a subfamily of topology, this G will be a member of the box topology. But the question is, whether this G is a member of product topology. The answer is no because for the product topology, the basis elements look like as $\prod_{i \in I} G_i, G_i \in \mathcal{T}_i$, for all $i \in I$ and G_i equals X_i except for finitely many values of i. But the G we have taken here, this is the product of G_i, where none of G_i equals to X_i. So, what have we seen, we have discussed two topologies on an arbitrary product of sets. One was generated by a basis and one was generated by a sub-basis. In the case when I is a finite set, both the topologies coincide, that is, both are equal. Even, we have seen that the product topology is strictly coarser than the box topology.

These are the references.

That's all from this lecture. Thank you.