Course Name: Essentials of Topology
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Week: 04
Lecture: 01

Welcome to Lecture 20 on Essentials of Topology.

This lecture is towards the study of subspaces, which provide a way for the creation of new topologies. Let us see what the concept is. The idea is, if we are having a topological space $(X, \mathcal{T})$ and if we are taking a subset Y of X. What are the topologies we can put on Y? The answer is that we can put a number of topologies on Y. At least two are well-known, that is, discrete topology as well as indiscrete topology. But our interest is to put a topology on Y with the help of the topology given on X. So, let us denote this one as $\mathcal{T}_Y$. The question is, how do we define this $\mathcal{T}_Y$? The answer is, if we are taking $\mathcal{T}_Y = \{Y \cap G : G \in \mathcal{T}\}$, we can show that this is a topology. Also, this is a new topology created with the help of the original topology, and our interest is only in this topology here. So, formally, let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. Then $\mathcal{T}_Y = \{Y \cap G : G \in \mathcal{T}\}$ is a topology on Y. Let us justify it. First, note that $\emptyset \in \mathcal{T}_Y$, because $\emptyset = Y \cap \emptyset, \emptyset \in \mathcal{T}$. Similarly, $Y \in \mathcal{T}_Y$, because $Y = Y \cap X, X \in \mathcal{T}$.

Moving ahead, let us take some finite number of sets in $\mathcal{T}_Y$, that is $H_1, H_2, ..., H_n \in \mathcal{T}_Y$. Our motive is to justify that $H_1 \cap H_2 \cap ... \cap H_n \in \mathcal{T}_Y$. If we want to justify it, see if $H_1, H_2, ..., H_n \in \mathcal{T}_Y$, what will happen? This H_1 can be expressed as $Y \cap G_1$, H_2 can be expressed as $Y \cap G_2$, ..., and H_n can be expressed as $Y \cap G_n$. Note that $G_1, G_2, ..., G_n$ are members of the topology $\mathcal{T}$. If we are computing $H_1 \cap H_2 \cap ... \cap H_n$, that will be nothing but Y$\cap(G_1 \cap G_2 \cap ... \cap G_n)$. This can be written as $Y \cap G$, where $G = G_1 \cap G_2 \cap ... \cap G_n$. Note that this G also belongs to $\mathcal{T}$, and therefore $H_1 \cap H_2 \cap ... \cap H_n$ is a member of $\mathcal{T}_Y$.

Moving ahead, let us take an indexed family of sets, that is $\{H_i : i \in I\}$, where every H_i is in $\mathcal{T}_Y$. Our motive is to justify that $\cup\{H_i : i \in I\}$, this also belongs to $\mathcal{T}_Y$. If we want to show it, again see the structure. Because every H_i is in $\mathcal{T}_Y$. So, what will happen? This H_i can be written as $Y \cap G_i$, where $G_i \in \mathcal{T}$. Now, if we are computing $\cup\{H_i : i \in I\}$, this can be written as $\cup\{Y \cap G_i : i \in I\}$, or this can be expressed as $Y \cap (\cup\{G_i : i \in I\})$, or this is something like $Y \cap G$, where $G = \cup\{G_i : i \in I\}$. Note that, if G_i is a member of $\mathcal{T}$, $i \in I$, so what will happen? Here $G = \cup\{G_i : i \in I\}$ is also a member of $\mathcal{T}$. Therefore, $\cup\{H_i : i \in I\} \in \mathcal{T}_Y$. Thus, this $\mathcal{T}_Y$ is a topology on Y.

Finally, we have these things. If we are having a topological space $(X, \mathcal{T})$ and a subset Y of X. Then we have seen that this $\mathcal{T}_Y = \{Y \cap G : G \in \mathcal{T}\}$, is a topology. We say this topology the relative topology or subspace topology. Note that this $(Y, \mathcal{T}_Y)$ is called a subspace of topological space $(X, \mathcal{T})$.

Let us take some examples of relative topology. If we are having a set X, let us take this as $\{a, b, c, d\}$. Put a topology $\mathcal{T}$ on it, that is, $\mathcal{T} = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$. If we are taking a subset $Y = \{b, c, d\}$ of X, then the relative topology on Y will be given by $\mathcal{T}_Y = \{\emptyset, Y, \{b, c\}\}$.

Moving ahead, let us take another example. This is a well-known example of Euclidean topology on the real line. Let us take a subset $Y = [0, 1]$ of $\mathbb{R}$. If we are constructing relative topology on Y, how that will look like? Note that whenever we are finding out this $\mathcal{T}_Y$, this is nothing but $\{Y \cap G : G \in \mathcal{T}\}$. One thing is clear that if we are taking $Y \cap \mathbb{R}$, that will be the closed interval $[0, 1]$. Thus $[0, 1] \in \mathcal{T}_Y$. Also, $\emptyset \in \mathcal{T}_Y$. What others? Note that the open sets in this Euclidean topology are the union of open intervals, and as we know that all the open intervals are open sets in the Euclidean topology. So, we can conclude that $[0, a), (a, b), (a, 1]$ are also members of $\mathcal{T}_Y$, where $0 < a < b < 1, a, b \in \mathbb{R}$.

These structures provide some interesting and important facts about the relative topology. The first thing is, if we are looking this closed interval $[0, 1]$. This closed interval $[0, 1]$ is a member of the relative topology. Meaning is, this closed interval $[0, 1]$ is $\mathcal{T}_Y$-open. But note that $[0, 1]$ is not open in the Euclidean topology. Similarly, if we are looking at the interval $[0, a)$, this interval is not in the Euclidean topology, but it is open with respect to the relative topology. Also, if we are looking for (a, b), this is open in this relative topology, and further, this is also open in the Euclidean topology.

One thing is clear from here. When we are trying to compare this Euclidean topology or relative topology or in other sense, if we are having a topological space $(X, \mathcal{T})$, a subset, that is a non-empty subset Y of X, and we are talking about $(Y, \mathcal{T}_Y)$. So, whether $\mathcal{T}_Y$ will be coarser than $\mathcal{T}$ or $\mathcal{T}_Y$ will be finer than $\mathcal{T}$, or there is no comparison between $\mathcal{T}_Y$ as well as $\mathcal{T}$. The answer is from this example. It is clear that we cannot compare $\mathcal{T}_Y$ and $\mathcal{T}$.

Moving ahead, let us take another example, and again, the example is from Euclidean topology. If we are taking the set of real numbers with Euclidean topology, let us take the set of integers, which is obviously a subset of the set of real numbers. If we want to find out what is the relative topology on this set of integers. Then it is clear from here that if we are taking any integer n, then how or what about this $\mathbb{Z}$ intersection $(n-1, n+1)$. Is it singleton set $\{n\}$? The answer is yes. Meaning is, for all $n \in \mathbb{Z}$, the singleton set $\{n\}$ will always be a member of $\mathcal{T}_{\mathbb{Z}}$. So, what can we conclude about this $\mathcal{T}_{\mathbb{Z}}$? Note that this $\mathcal{T}_{\mathbb{Z}}$ is nothing, but this is a discrete topology.

Again, we have one more observation here. Note that what we are talking about here, that is, we are talking about this type of structure, that is, $(X, \mathcal{T})$. We are taking a subset Y of X, and obviously, this Y is non-empty. We are talking about the subspace of $(X, \mathcal{T})$. Note that the original topology is not discrete, but the relative topology, what we are constructing here, that becomes a discrete topology. Also, this is true in the case of indiscrete topology. For example, if we are taking X as three elements set, that is $\{a, b, c\}$, let us put a topology $\mathcal{T}$ on it, that is $\{\emptyset, X, \{b, c\}\}$. Now, we are taking Y as this $\{b, c\}$, and we are constructing the relative topology on Y. That will be $\mathcal{T}_Y = \{\emptyset, Y\}$. Again, note here that the topology $\mathcal{T}$ is not indiscrete, but the relative topology is indiscrete. It is clear from this example that even though the given topological space is neither discrete nor indiscrete, the relative topological spaces may be discrete as well as indiscrete.

Moving ahead, if we are thinking in converse direction, that is, if the topological space $(X, \mathcal{T})$, this is, for example, indiscrete. What about the relative topology? That is, what about its subspace? The answer is, in this case, this will also be indiscrete. Why? Let us see it. Because this $\mathcal{T}$ is nothing but a collection of the empty set as well as X. Now, if we are constructing this $\mathcal{T}_Y$, that is $\{\emptyset, Y\}$. Meaning is, the relative topology is indiscrete. Even if we are taking this $(X, \mathcal{T})$, that is, this topology is discrete. Let us take a subset Y of X. What about this $\mathcal{T}_Y$? $\mathcal{T}_Y = \{Y \cap G : G \in \mathcal{T}\} = \{Y \cap G : G \subseteq X\} = \{G' : G' \subseteq Y\} = P(Y)$. So, we can conclude that this topology is discrete.

Thus, finally, we have the conclusion that the relative topology of an indiscrete topology is indiscrete. Similarly, the relative topology of a discrete topology is discrete. But the converse is not true, which we have already seen.

Moving ahead, let us discuss some results on this relative topology and subspaces. The first one is a characterization of closed sets in relative topology. If we recall, we have seen that if this H is a member of $\mathcal{T}_Y$, that is the relative topology, then this H can be written as $Y \cap G$, where $G \in \mathcal{T}$. Meaning is, the open sets in relative topology can be expressed as Y intersection some open set of the original topology. The question is, whether we can get or have some similar observation in the case of closed sets. The answer is yes, and that is characterized by this result. The result is: Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. Then $F \subseteq Y$ is $\mathcal{T}_Y$-closed iff $F = Y \cap F'$, for some $\mathcal{T}$-closed set F'. This is a similar observation, a similar concept to what we have seen in the case of open sets. So, let us justify it. First we are assuming that F is $\mathcal{T}_Y$-closed. Now, if F is $\mathcal{T}_Y$-closed, what will happen? By definition, obviously, its complement is $\mathcal{T}_Y$-open, and if this is $\mathcal{T}_Y$-open, then F^c can be expressed as $Y \cap G$. Note that this G is from $\mathcal{T}$. If we are taking the complement, then this F can be expressed as $Y - (Y \cap G)$. Note that G is in $\mathcal{T}$, or this can be expressed as $Y \cap (Y \cap G)^c$, G is from $\mathcal{T}$. Thus $F = Y \cap (Y^c \cup G^c) = (Y \cap Y^c) \cup (Y \cap G^c) = \emptyset \cup (Y \cap G^c)$. Thus, F is precisely $Y \cap G^c$, and G is in $\mathcal{T}$. This is clear from here that F can be written as $Y \cap F'$, where $F' = G^c$. Note that G is $\mathcal{T}$-open, so F' is $\mathcal{T}$-closed. Thus, F can be expressed as $Y \cap F'$, where F' is a $\mathcal{T}$-closed set.

Conversely, if $F = Y \cap F'$, let us show that F is $\mathcal{T}_Y$-closed. Now, try to justify that F^c is $\mathcal{T}_Y$-open. So, if F can be written as $Y \cap F'$, let us see what about the complement of F. Its complement is nothing but $Y - (Y \cap F')$. If we are simplifying, this is nothing but $Y \cap (Y \cap F')^c$, or this is $(Y \cap Y^c) \cup (Y \cap F'^c)$. So, what we have seen here is that F^c can be written as $Y \cap F'^c$. But note that what is F'? F' is $\mathcal{T}$- closed, and therefore, F'^c is $\mathcal{T}$-open. If this is $\mathcal{T}$-open, then we can say that F^c is $\mathcal{T}_Y$-open or F is $\mathcal{T}_Y$-closed. That's all about this result.

Moving to the next result, this result is regarding the basis for relative topology. We have already seen that in the case of open sets and closed sets for relative topology, the structures look like of the form Y intersection open set or Y intersection closed set. The question is, whether a similar observation can be given for the basis for relative topology. The answer is yes, and the result is here. Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. If $\mathcal{B}$ is a basis for topology $\mathcal{T}$, then $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$ is basis for $\mathcal{T}_Y$. The justification is simple. For example, if we are taking any open set, that is $\mathcal{T}_Y$ open set H, and let us take $y \in H$. Note that because H is $\mathcal{T}_Y$-open, it can be written as $Y \cap G$. So, what we have with us that $y \in Y \cap G$, note that this $G \in \mathcal{T}$. Also, if $y \in Y \cap G$, obviously $y \in G$. Note that this G is coming from topology $\mathcal{T}$. By using the definition of basis, because $\mathcal{B}$ is a basis for topology $\mathcal{T}$, there exists some $B \in \mathcal{B}$ with the property that $y \in B$, and that is a subset of G, or we can also write that $y \in Y \cap B \subseteq Y \cap G$. So, if we recall the definition of a basis for topology, it confirms that $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$ is a basis for $(Y, \mathcal{T}_Y)$.

Moving ahead, this result is regarding the structure of the neighborhood in the relative topology. Let $(Y, \mathcal{T}_Y)$ be a subspace of a topological space $(X, \mathcal{T})$. Then $N \subseteq Y$ is a $\mathcal{T}_Y$-nbd of $y \in Y$ iff there exists a $\mathcal{T}$-nbd M of y such that $N = Y \cap M$. This expression is similar to what we have seen in the case of open sets, closed sets, as well as basis. So, let us prove this result. Begin with here, let us assume that N is $\mathcal{T}_Y$-neighborhood of y, and if we are assuming it, then by the definition of the neighborhood, there exists a $\mathcal{T}_Y$-open set. Let us take that open set be H such that $y \in H \subseteq N$. But note that this H can be expressed as $Y \cap G$, and G is $\mathcal{T}$-open. So, this can be written as $y \in Y \cap G$, which is a subset of N. Our motive is to construct an M. The natural choice for M is, if we are taking M as $G \cup N$. What we can write from here is that y belongs to G, which is a subset of $G \cup N$, that is, M itself. So, from this expression, it is clear that M is a $\mathcal{T}$-neighborhood of y. But the question remains to justify whether $N = Y \cap M$. Let us see it. If we are finding out $Y \cap M$, that will be $Y \cap (G \cup N)$, or that will be $(Y \cap G) \cup (Y \cap N) = H \cup (Y \cap N) = H \cup N = N$. So, it proves the first part.

Coming to the converse part of the result that if there exists a $\mathcal{T}$-neighborhood M of y such that $N = Y \cap M$, then let us prove that N is a $\mathcal{T}_Y$-neighborhood of y. So, what we are doing is assuming this part. Note that if M is a $\mathcal{T}$-neighborhood of y, then there exists a $\mathcal{T}$-open set G such that we can write $y \in G \subseteq M$, or $y \in Y \cap G \subseteq Y \cap M$, or $y \in Y \cap G \subseteq N$, because $N = Y \cap M$. Also, note that if G is $\mathcal{T}$-open, what about this $Y \cap G$? That is nothing but $\mathcal{T}_Y$-open, or from here we can conclude that N is $\mathcal{T}_Y$-neighborhood of y.

These are the references.

That's all from this lecture. Thank you.