Course Name: Essentials of Topology
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Welcome to the second lecture on Essentials of topology. As discussed in the previous lecture, the ideas from sets and functions play a key role in the study of topology. In this lecture, we will recall the concepts associated with sets and functions. Accordingly, the topics which we will cover in this lecture are sets, operations on sets, and functions. Even these concepts are well known. Still, let us have a look at these.

Beginning with the concept of a set. As it is well known that a set is a collection of objects, also known as elements or members. There are different ways to specify a set. For example, a set consisting of four elements $\{a, b, c, d\}$ can be specified by the set A. Similarly, the set consisting of integers from 1 to 50 can be specified by using set B. Similarly, the set consisting of integers greater than 10 can be given by the set C, and a set consisting of odd integers can be given by the set D, and it can be read as "this is a collection of all x such that x is an odd integer".

Moving ahead, these are some standard notations. The first one is for $x \in A$. It means that x is a member of set A while the second one is for $x \notin A$; that is, x is not a member of set A. The next is the concept of empty set and singleton set. These two are well known. An empty set means a set having no element, while a singleton set means a set having only one element. These are some of the useful notations for sets which are standard. The first one is $\mathbb{N}$, the set of natural numbers. The second one is $\mathbb{Z}$, the set of integers. The third one is $\mathbb{Q}$, the set of rational numbers. The last one is $\mathbb{R}$, the set of real numbers.

Even there are some well-known subsets of set of real numbers called intervals. The first one is known as an open interval. Second and third, these are called semi-open intervals. The last one is a closed interval. This interval is defined as a collection of all those real numbers x such that x lies between a and b, that is, $a < x < b$. When we are talking about the semi-open interval open from the right, this is a collection of all real numbers such that $a \leq x < b$.

The semi-open interval open from the left is given as a collection of all real numbers such that $a < x \leq b$. Finally, the closed interval $[a, b]$ is a collection of all real numbers such that $a \leq x \leq b$.

Moving to the next concept, for given two sets, A and B, we call that A is a subset of B, and we use this particular notation ($\subseteq$) if each element of A is also an element of B, or formally, for all $x \in A$, $x \in B$. We say that two sets, A and B, are equal if $A \subseteq B$ and $B \subseteq A$. Similar to the concept of subset, if we remove the condition of equality, that is, $A \subseteq B$ but $A \neq B$, in this case, we say that A is a proper subset of B, and we use this notation here ($\subset$).

The concept of subsets leads us to define a new notion that is known as the power set of A. For a given set A, the power set of A is a collection of all those set B, such that $B \subseteq A$. It is to be noted here that if A has n elements, this notation ($|A|$) is for number of elements in A; then we can talk about the number of elements in the power set of A, and that is 2^n. Let us discuss some operations on sets. The first such operation is the concept of union. For given two sets, A and B, $A \cup B$ is defined as a collection of all those x such that $x \in A$ or $x \in B$, and in terms of the Venn diagram, the shaded region represents $A \cup B$. It is to be noted here that whenever we are saying that $x \in A \cup B$, we call it that $x \in A$ or $x \in B$. But whenever we say that $x \notin A \cup B$, it means that $x \notin A$ and $x \notin B$.

Moving to the second operation, that is the concept of the intersection of two sets. Again, for given two sets, A and B, their intersection is a collection of all elements which are common to both A as well as B, that is, $A \cap B$ is a collection of all those x such that $x \in A$ and $x \in B$. In this Venn diagram, the shaded region represents $A \cap B$. Similar to the concept of union, again, it is important to note that whenever we talk about $x \in A \cap B$, we say that $x \in A$ and $x \in B$. But whenever we are saying that $x \notin A \cap B$, it means that $x \notin A$ or $x \notin B$.

Moving to the next concept, that is, the concept of set difference. For given two sets, A and B, their difference, read as A minus of B (denoted as $A - B$), is a collection of all elements of A such that $x \notin B$. From this Venn diagram, it is clear that if we have two sets, A and B, this shaded region represents

$A - B$. It should be noted that, this is not the difference of numbers.

Coming to the next concept, that is, the concept of De Morgan's law. De Morgan's law is based on notions of union, intersection, plus one more concept, the complement of a set in a given set. So, if we are having a set X and a subset A of X, then the complement of A in X (denoted as, A^c) is nothing but a collection of all elements of X such that $x \notin A$. It is clear that this can also be read as $X - A$. In terms of the Venn diagram, if X is the set and $A \subseteq X$, then this shaded region represents A^c. Continuing to De Morgan's law. De Morgan's law states that the complement of $A \cup B$ is the same as $A^c \cap B^c$. Dual to it, the complement of A intersection B is the same as $A^c \cup B^c$.

The next concept is the Cartesian product of two sets. If we have two sets, A_1 and A_2, their Cartesian product is denoted as $A_1 \times A_2$. This is a collection of all ordered pairs (a_1, a_2) such that $a_1 \in A_1$ and $a_2 \in A_2$. For example, if we are taking set A_1, this is a singleton set $\{1\}$, and set A_2 is a set consisting of two elements, $\{a, b\}$. Then $A_1 \times A_2$ is a collection of ordered pairs, $(1, a)$ and $(1, b)$. We can also talk about the Cartesian product of A_2 and A_1, which is a collection of elements $(a, 1)$ and $(b, 1)$. It is to be noted here that $A_1 \times A_2$ is not equal to $A_2 \times A_1$. We will take some more examples, wait for it.

It is simple to justify that $A_1 \times A_2 = \emptyset$ if and only if either $A_1 = \emptyset$ or $A_2 = \emptyset$. Just think about it. Moving ahead, we have already seen that the Cartesian product of two sets may not be commutative, but the question is, when $A_1 \times A_2$ can be equal to $A_2 \times A_1$? The answer is when $A_1 = A_2$, and the result is stated here.

The Cartesian product of two sets can be generalized to n number of sets. For example, for any $n > 2$, if we want to find out the Cartesian product of n sets $A_1, A_2, .., $ and A_n, that is given by the Cartesian product as $(A_1 \times A_2 \times ... \times A_{n-1}) \times A_n$. This is based on the concept of induction. If $A_1, A_2, ..,$ and A_n are sets of real numbers, then this Cartesian product is denoted by $\mathbb{R}^n$. For $n = 2$, that is $\mathbb{R}^2$, we are talking about plane; for $n = 3$, it becomes $\mathbb{R}^3$, and so on. So these are the standard notations which we will use.

Let us take some diagrams regarding the Cartesian product of two sets. If we are looking at the first one, the first one is the real numbers from 0 to 1 and this is from 0 to 3. This is the Cartesian product of what sets? The answer is: this is the Cartesian product of closed intervals $[0,1]$, and $[0,3]$. So this shaded region is the Cartesian product of closed intervals $[0,1]$ and $[0,3]$.

If this, we are representing the real line; this is the set of real numbers with $[0,3]$ as a closed interval. If we are coming to this picture, and we want to see what the Cartesian product is represented by this line parallel to y-axis and passing through $x = 3$. The answer is, this is nothing but $\{3\} \times \mathbb{R}$, which is precisely all $(x, y) \in R^2$ such that x is fixed, that is 3, and y is any real number, or in another way this is nothing, but this is $(3, y) \in R^2$ such that y is any real number. In the same fashion, one can write about the set of $\mathbb{R} \times [0,3]$ or $[0,1] \times [0,3]$.

Moving to the next concept, let us recall the notion of functions. For two sets X and Y, a function f from X to Y is denoted like this one ($f : X \to Y$) and defined as a rule that to each element $x \in X$, it associates a unique element $f(x) \in Y$. This X has a special name, that is X is called the domain of f, Y the co-domain of f; x is called the pre-image of $f(x)$ and $f(x)$ is called the image of x under f. Let us take some examples. There are three diagrams on the screen. The first one, if we see and if we are taking this is a rule f, what is it doing? This is sending the first element on here, the second element here, and the third element here, but what about this one? There is no element of Y associated with this particular element. So, this is not a function.

Coming to the second example, let us take a rule f from X to Y, this is sending the first element to the first one, this is sending the second element to the second as well as the third both, meaning is, that the value of $f(x)$ is not unique and therefore this is also not a function. Coming to the next one, let us take rule f, which is sending the first element of X to the first element of Y, this is the second element of X to the second element of Y, and the third element is here, this is also going on the second element of Y. Two things are clear here that this rule is assigning some elements of Y to every element of X, and the second one is that this $f(x)$ is always unique, corresponding to each x, and therefore this is an example of a function.

Formally, for the first one, let us take $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x + 5$; this is a function. Geometrically, this is nothing but an equation of line. $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$; this is also a function; $f(x) = e^x$, that is an exponential function, $f(x) = |x|$, that is, the modulus function, so these four are some of the well-known examples of a function.

Let us see their graph. The first one is the graph of the function $f(x) = 2x+5$, but we can see that if we are drawing a vertical line, what is happening? This vertical line on the graph of f never crosses the graph at more than one point; this is to be noted. Similarly, the same thing is happening if we are taking $f(x) = x^2$, and again, we are drawing some vertical lines here. This is never crossing the graph at more than one point.

Similarly, we are coming to the modulus function, whenever we are drawing a vertical line, this never crosses the graph at more than one point, and the same thing is happening in the case of exponential function, meaning is that, if we are taking any function a vertical line on the graph of f never crosses the graph in more than one point, when function is defined from $\mathbb{R}$ to $\mathbb{R}$. For example, if we are taking another one, let us plot a curve like a circle, and the question is whether this circle represents a function. The answer is no. Why? Because, if we are drawing a vertical line, this is intersecting the curve at two points, and therefore, this does not represent a function.

Moving to the next, we will discuss some particular types of functions. Such functions are known as one-to-one or injective functions. A function $f : X \to Y$ is called one-to-one or injective, if we are taking two elements of the domain x_1 and x_2; in case they are not equal, $f(x_1)$ should not equal to $f(x_2)$, or equivalently if $f(x_1) = f(x_2)$ then $x_1 = x_2$. It is clear from here that a function is one-one if distinct elements have distinct images. For example, $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x + 5$. This is an injective function or one-one function because if we are taking $f(x_1) = f(x_2)$. Then $2x_1 + 5 = 2x_2 + 5$, or by simplifying it, we will get $x_1 = x_2$. Therefore, $f(x) = 2x + 5$ is an injective function.

Let us take another function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Note that this function is not injective because it is sending distinct elements of $\mathbb{R}$ to the same element of $\mathbb{R}$. For example, if we are taking $f(-2)$, that will be 4, and if we are computing $f(2)$, that will always be 4. Meaning is that, images of distinct elements of the domain are not distinct, that's why this is not an injective function.

But if we are taking a function by changing the domain of the previous function, that is, f is defined from this interval to the set of real numbers such that $f(x) = x^2$. In this case, this function becomes injective, and this is because of the restrictive domain of the function. In this case, if we are taking $f(x_1) = f(x_2)$, in this case, $x_1^2 = x_2^2$, or $x_1 = x_2$, because $0 \leq x_1, x_2 < \infty$. Let us have a look at the graph of some of the functions. The first one is, this is $f(x) = 2x + 5$, this is $f(x) = x^2$, when f is defined from $\mathbb{R}$ to $\mathbb{R}$; and this is $f(x) = x^2$, when f is defined from $[0, \infty)$ to $\mathbb{R}$.

If we observe in all three cases, what is going on? If we are taking any horizontal line, and how is the horizontal line intersecting the graph. So, see in the first case, if we are taking any horizontal line this line is intersecting the graph at a single point. Similarly, if we are drawing some horizontal lines, then this is also intersecting the graph at a point. But if we are going with the same concept, in the case of this function, note that here, the intersection of each horizontal line is at two points. So there is a simple characterization of the concept of injective function, that, $f : \mathbb{R} \to \mathbb{R}$ is injective if and only if every horizontal line intersects the graph of f at most one point. This is important.

Coming to the concept of onto or surjective function. A function $f : X \to Y$ is called a surjective function or onto function, if for all $y \in Y$, that is, for every y belongs to co-domain, there exists $x \in X$, that is, x is in domain with the condition that $f(x) = y$. In a nutshell, we can say that each element of the co-domain has a pre-image. Let us take some examples of onto or surjective functions. So the first one $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) = 2x + 5$, is an onto function, because if we are taking any $y \in \mathbb{R}$, we can always find x, which is nothing but $(y - 5)/2 \in \mathbb{R}$ such that if we are computing $f(x)$, that will be $2[(y-5)/2]+5$, that becomes y. Therefore, this function is an onto function.

If we are looking at the second example, $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) = x^2$. Note that this function is not surjective because if we are taking $y = -9$, we cannot find out any $x \in \mathbb{R}$ such that $f(x) = -9$. But if we are making a simple change in the co-domain of the function that is an interval $[0, \infty)$ such that $f(x) = x^2$, one can check that this is an onto function.

In terms of a graph, let us take the first one, this is a line $y = 2x + 5$, and the second is $y = x^2$ or $f(x) = x^2$. Again, if we are drawing a horizontal line to the first one, this is intersecting the graph of $y = 2x + 5$ at one point. If we are drawing lines or horizontal lines in the case of $f(x) = x^2$, then this is intersecting the graph of $f(x) = x^2$ at two points. So from these two graphs, what we can conclude that $f : \mathbb{R} \to \mathbb{R}$ is surjective if and only if every horizontal line intersects the graph of f at least once.

Moving to the next concept, that is the concept of a bijective function. We say that a function $f : X \to Y$ is bijective, if it is both injective as well as surjective. For example, let us take a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x + 5$. This function is a bijective function, because we have already seen that this function is both injective as well as surjective. Let us take another example, that is $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Then, this function is not bijective. Why? Just think about it. Even we have already seen it. We also have a conclusion on a function $f : \mathbb{R} \to \mathbb{R}$: when will this function be bijective? The answer is if and only if every horizontal line intersects the graph of f exactly once.

This conclusion is based on the idea that a bijective function means it is both injective as well as surjective, and we have already seen that in one case, the intersection is at most one point, and in another case, the intersection is at least one point and combining both we get the conclusion about exactly once.

These are the references.

That's all from this lecture. Thank you.