Course Name: Essentials of Topology
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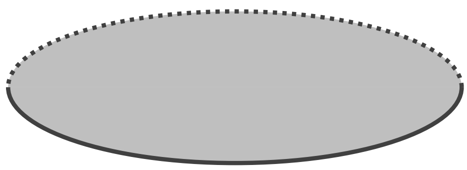
Welcome to Lecture 19 on Essentials of Topology.

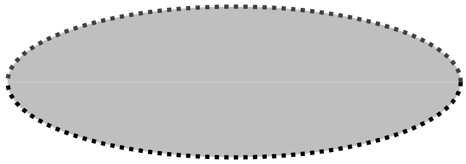
In this lecture, we will study the concepts of interior and boundary of sets. Similar to the concept of closure of sets, we will see here that the interior will help us to provide a characterization of open sets. Also, it will give a notion of the largest open set contained in a set. Beginning with the concept of the interior of sets, this is similar to the concept of closure of sets, where we have seen that how to find out the smallest closed set containing a set. What we are going to see, how one can find out the largest open set inside a given set.

Begin with the definition. Let $(X, \mathcal{T})$ be a topological space and a subset A of X. Then the interior of A is denoted like $Int(A)$ or A°. This is defined as $\cup\{G \subseteq A : G \text{ is an open set}\}$. It means that if we are having a set X and we are having a subset A, whenever we are trying to find out A°, try to take the union of all open sets inside A. What we will get is nothing but the interior of the set.

From the definition itself, these are some observations. The first one is that A° is an open set because this is the union of open sets. The second one is, A° will always be a subset of A. From the definition, this is clear, as it is the union of open sets inside A. The third one is, we can say that A° is the largest open set inside A. Moving ahead, let us visualize the concept of the interior of sets in $\mathbb{R}^2$. For example, if this is a set A.



Then the A° will look this way: that is an open set contained in the set, and note that this is the largest open set inside A.



Let us take some of the examples. For example, if we are taking X as a set $\{a, b, c, d, e\}$. Take a topology, $\mathcal{T}$, that is $\mathcal{T} = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}, \{a, b, c\}\}$. It can be seen that this is a topology on X. Now, if we are taking a set A which is given as $\{a, d, e\}$ and we want to find out A°. So, we have to find out the union of all open sets inside A. Now, see what the open sets contained in A are. Obviously, the empty set is here, and if we are looking for the rest, none of the open sets are subsets of A. That's why A° will be the empty set. But if we are taking another set B. For example, we are taking B something like $\{a, b, d\}$. Let us try to compute B°, that is, the union of all open sets inside B. So, the first one is the empty set. The second is, note that this two elements set $\{a, b\}$ is a subset of $\{a, b, d\}$. Even the singleton set $\{b\}$ is a subset of $\{a, b, d\}$. That's why B° will be nothing but this $\{a, b\}$.

Moving ahead, let us take some other examples. Why not $\mathbb{R}$ with this Euclidean topology? If we are taking a set A as an open interval $(2, 3)$. What about its interior? The interior will be nothing but this open interval itself because this is the largest open set inside A. But if we are taking A as a closed interval $[2, 3]$. What about A°? Note that this will still open interval $(2, 3)$ because this is the largest open set inside the closed interval $[2, 3]$. Moving ahead, why not let us take A as this semi-open interval $[2, 3)$. What about its interior? Again, the interior will be $(2, 3)$. It is to be noted here that whenever we are taking A as semi-open interval $[2, 3)$, its interior is this open interval $(2, 3)$. But if we are changing topology, that is, if we are taking $\mathbb{R}$ with the lower limit topology and we are computing A°, when A is this semi-open interval $[2, 3)$. Note that this interior will be this semi-open interval itself, because this semi-open interval is open in the lower limit topology. Meaning is, the concept of this interior depends on the nature of the topology with which you are working. Even on any set X, if we are taking the discrete topology, the question is, what about the interior of a subset? So, if we are taking any subset of X, the interior will always be the set itself because every set is open. That's why the interior of A will always be equal to A.

Coming to this characterization of the open sets by using the concept of the interior, what we say is that if $(X, \mathcal{T})$ is a topological space and $A \subseteq X$, then A is open if and only if $A^\circ = A$. The first one is simple. Whenever we are taking $A^\circ = A$, note that A° is open, and therefore, A is also open. That's why this is a simple one. Conversely, let A be open. Assume it, and let us try to justify that $A^\circ = A$. Note that A° is always a subset of A, and A is also a subset of A. What is A°? We have seen that this is the largest open set inside A, and this is also given that A is open. From the set theory, A is the largest set, which is open inside A. So, from here, we can conclude that A° is nothing but A itself.

One more thing is clear from here that if we are taking x as an element of A°, note that $A^\circ = \cup\{G \subseteq A : G \text{ is an open set}\}$. So, if $x \in A^\circ$, we say that x is an interior point of A. Now, what will happen if x is an interior point of A? So, from this definition, x will belong to some G. This is because x is an element of $\cup\{G \subseteq A : G \text{ is an open set}\}$, and note that G is a subset of A. So, what can we conclude? We can conclude that A is a neighborhood of x. Similarly, if we are assuming that A is a neighborhood of x, then what will happen? There exists an open set G with the property that $x \in G \subseteq A$. Note that this G is a subset of A, and we are taking the union of all such G. So, from here, we can also say that $x \in \cup\{G \subseteq A : G \text{ is an open set}\}$. If this is the case, $x \in A^\circ$. So, what conclusion do we have here? If x is an interior point of A, then A is a neighborhood of x, and conversely, if A is a neighborhood of x, x is an interior point of A. Finally, we have a result with us: A is a neighborhood of $x \in X$ if and only if x is an interior point of A. This is a simple characterization.

Moving ahead, let us see some results on the interior of the sets. Beginning with, if we are having a topological space $(X, \mathcal{T})$ and two subsets A and B of X, then the first one is: $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$. If we want to justify this result, it's a simple one. If we are taking $x \in A^\circ$, what does it mean? It means that A is a neighborhood of x. We have already seen this characterization. It means that B is also a neighborhood of x because B is a superset of A. Therefore, we can say that $x \in B^\circ$. That's all about the first one. Coming to the second one, i.e., $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$: this directly follows from the first because we know that $A, B \subseteq A \cup B$. So, $A^\circ, B^\circ \subseteq (A \cup B)^\circ$. If we are combining these two, $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$. But note that equality may not hold in general. Let us see an example. If we are taking real line with the Euclidean topology. Let us take a set A, that is the closed interval $[2, 3]$. Take a set B, that is a closed interval $[3, 4]$. If we are looking for $A \cup B$, it is nothing but the closed interval $[2, 4]$. If we are computing A°, we have already seen that this is open interval $(2, 3)$. What about B°? That is an open interval $(3, 4)$. But if we are computing $(A \cup B)^\circ$, that will be open interval $(2, 4)$. Now, what about $A^\circ \cup B^\circ$? That is $(2, 3) \cup (3, 4)$. This is the union of two open intervals, and note that this is not equal to an open interval $(2, 4)$. Therefore, equality may not hold in general.

Coming to the next one, i.e., $(A \cap B)^\circ = A^\circ \cap B^\circ$. In order to justify this result, we have to show that $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. Also, $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$. The first can be directly obtained because we know that $A \cap B$ is always a subset of A, and $A \cap B$ is also a subset of B. From these two, we can conclude that $(A \cap B)^\circ \subseteq A^\circ$. Also, $(A \cap B)^\circ \subseteq B^\circ$. Therefore, $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. Coming to the second one, here, if we are taking $x \in A^\circ \cap B^\circ$, then $x \in A^\circ$, and x is also an element of B°. If we are using the concept of neighborhoods, from here, we can say that A is a neighborhood of x. Similarly, we can deduce here that B is a neighborhood of x. We have already seen that the intersection of two neighborhoods is a neighborhood. So, from these two, we can conclude that $A \cap B$ is a neighborhood of x. Also, if $A \cap B$ is a neighborhood of x, we can say that x is an interior point of $A \cap B$. That is, $x \in (A \cap B)^\circ$, and that's all about this result.

Moving ahead, let us see the relationship between closure as well as the interior of a set. These are two simple characterizations given in terms of the complement of closure and the complement of the interior. If we are having a topological space $(X, \mathcal{T})$ and a subset A of X, the first result states that $(X - A)^\circ = X - \bar{A}$. While the second states that $\overline{X - A} = X - A^\circ$. If we can prove one, the proof of the second is similar to that. So, let us establish the first one. If we are looking for the first one, we have to justify two things, that is $(X - A)^\circ \subseteq X - \bar{A}$. Also, for the second one, we have to justify that $X - \bar{A} \subseteq (X - A)^\circ$. The second one can be deduced by using the concept of the interior, that is, the definition of interior itself, because we know that $\bar{A}$ is a closed set. Also, if this is closed, its complement is an open set. We know that A is a subset of $\bar{A}$, and if we are taking the complement of these two, A^c will be the superset of $X - \bar{A}$. Now, let us take the interior of both sides. So, this will be: $(X - A)^\circ$ is the superset of the interior of $X - \bar{A}$. But note that because $X - \bar{A}$ is an open set, therefore it's interior will be $X - \bar{A}$.

That's why, $X - \bar{A} \subseteq (X - A)^\circ$. Now, in order to justify the first one, let us take an element $x \in (X - A)^\circ$. Meaning is, $X - A$ is a neighborhood of x, and therefore, there exists an open set G such that this $x \in G \subseteq X - A$. Note that from here, this G is contained in A^c. Meaning is, we can say that $G \cap A$ will be an empty set. Also, note that $x \in G$. Meaning is, G is an open set and not only open set, it also contains x. Therefore, we can say from here, $(G - \{x\}) \cap A$ is an empty set, or x is not an element A'. Also, because x is an element of $X - A$ and $x \in G$. Meaning is, x does not belong to $A \cup A'$, or $x \in (A \cup A')^c$. That is, $x \in X - \bar{A}$. That's the proof of this result. That is, $(X - A)^\circ = X - \bar{A}$. In the same way, we can deduce the second result.

Moving ahead, coming to the next concept, that is, the concept of the boundary of sets. We have already studied the concept of closure of a set as well as the interior of a set. Actually, this boundary lies something in between closure and interior. Formally, this is defined in this way, that if we are having a topological space $(X, \mathcal{T})$ and a subset A of X. Then the boundary of A is nothing but, that is, given by $Bd(A) = Cl(A) - Int(A)$. One thing is clear from this definition because the interior of A is an open set, and if we are using some simple set theory, this is nothing but the closure of A intersection, the complement of the interior of A. Note that the interior of A is an open set. Therefore, the complement of the interior of A will be a closed set, and if this is closed, we already know that the closure of A is a closed set. That's why, this boundary of A is always a closed set.

Let us visualize this concept. For example, we are taking these pictures in $\mathbb{R}^2$. Let us take this as set A. We have already seen that this one represents the interior of A, and this one is representing the closure of A. If we are trying to find out the boundary of A, obviously this is the boundary of A, that is, $Cl(A) - Int(A)$.

Let us take some of the examples. For example, we are taking a set $X = \{a, b, c, d\}$. Let us take a topology $\mathcal{T}$ on it, that is, $\mathcal{T} = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$. Now, if we are taking a set A, something like, this is $\{b, c, d\}$. Let us see what the boundary of A means. If we are computing the interior of this set, the interior of the set is the union of empty set and other open sets which are inside $\{b, c, d\}$, that is $\{b, c\}$. So, the interior of A will be nothing but this two elements set $\{b, c\}$. If we are computing the closure of A, let us see what are the closed sets. The closed sets are: $X, \emptyset, \{b, c, d\}, \{a, d\}, \{d\}$. Now, we have to take the intersection of all closed sets containing this A, that is nothing but $X \cap \{b, c, d\}$, that will be A itself, that is $\{b, c, d\}$. Finally, if we are looking for the boundary, which we have already seen, this is $Cl(A) - Int(A)$. If we are computing this set, that will be the singleton set $\{d\}$. Let us take some more examples. For example, if we are taking $\mathbb{R}$ with the Euclidean topology. Also, let us take set A, that is an open interval, this is $(2, 3)$. What about the interior of A, which is nothing but the set itself? What about the closure of this set? Closure of this set is the closed interval $[2, 3]$. If we are looking for the boundary of A, that will be nothing but two points set that is $\{2, 3\}$. If we are taking another example in this Euclidean topology, let us take the set of rationals. We have already seen that this $\mathbb{Q}^\circ$ is an empty set. What about this $\bar{\mathbb{Q}}$? That is $\mathbb{R}$. So, what is the boundary? The boundary of $\mathbb{Q}$ is nothing but the set of real numbers. Meaning is, all the elements of $\mathbb{R}$ are in the boundary of $\mathbb{Q}$.

Moving ahead, these are some simple results related to the boundary of a set. For example, if we are having a topological space $(X, \mathcal{T})$ and having a subset A of X, then the first thing is: $Bd(A) = Cl(A) \cap Cl(X - A)$. It can be deduced by using the definition plus what we have already seen, that is, $Bd(A) = Cl(A) - Int(A)$. So, $Bd(A)$ can be written as $\bar{A} \cap (X - A^\circ)$. Also, we have already proven that this is $\bar{A} \cap \overline{(X - A)}$. The second result says that $Bd(A) \cap Int(A) = \emptyset$. It follows from the definition because by the definition of the boundary, $Bd(A) = Cl(A) - Int(A)$. Finally, coming to this one, i.e., $Bd(A) \cup Int(A) = Cl(A)$. This is simple to justify because if we are looking at the definition of the boundary of A, then what will happen? $Bd(A) \cup Int(A) = (Cl(A) - Int(A)) \cup Int(A) = (Cl(A) \cap (Int(A))^c) \cup Int(A)$. Now, by using the distributive property, it can be written as $(Cl(A) \cup Int(A)) \cap ((Int(A))^c \cup Int(A))$. If we are looking at the second one, note that this is nothing but the set X. Also, note that the interior of A is always contained in the closure of A because we have already seen that $A^\circ \subseteq A \subseteq \bar{A}$. So, $Bd(A) \cup Int(A)$ will be nothing but $\bar{A} \cap X$, and that's why this will be $\bar{A}$.

These are the references.

That's all from this lecture. Thank you.