Course Name: Essentials of Topology
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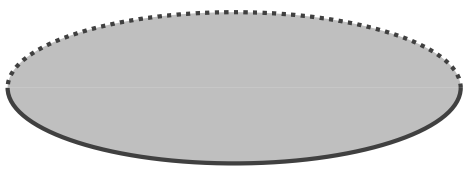
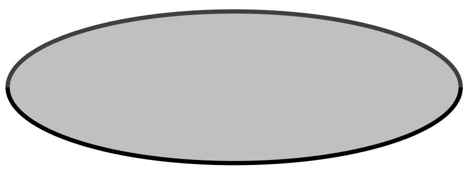
Welcome to Lecture 18 on Essentials of Topology.

In this lecture, we will study the concept of closure of sets. We will see that this concept provides the notion of the smallest closed set containing a set as well as the characterization of closed sets. Beginning with the concept of closure of a set. If we are having a topological space $(X, \mathcal{T})$ and a subset A of X, then the closure of A is denoted by $Cl(A)$ or $Cl(A)$, which is $\cap\{F \subseteq X : F$ is closed and $A \subseteq F\}$. From the definition, there are some simple observations. The first observation is that this $Cl(A)$ is a closed set because it is nothing but an arbitrary intersection of closed sets. That's why $Cl(A)$ is a closed set. Even, it is clear from here that A will always be a subset of $Cl(A)$ because this is the intersection of closed sets, and all the closed sets contain A. That's why A will always be a subset of $Cl(A)$. We can also observe that this $Cl(A)$ is the smallest closed set, and obviously, this is containing A. These are some simple observations from this definition.

Now, let us see this picture in $\mathbb{R}^2$. If this is set A, how does its closure look like?



The closure will look like this way. That is itself a closed set, and from the picture, it is clear that this is the smallest closed set which is containing A.



Let us take some of the examples. For example, if we are taking $X = \{a, b, c, d\}$. Also, let us take a topology $\mathcal{T}$ on X, that is, $\mathcal{T} = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$.

Note that this $\mathcal{T}$ is a topology on X. What about the closed sets with respect to this topology? So, closed sets are: X, $\emptyset$, $\{b, c, d\}$, $\{a, d\}$ and $\{d\}$. Now, if we are taking a set A, let us take this set as $\{a, b\}$, and if we want to find out the closure of this set, then the closure of this A is the intersection of all closed sets containing this $\{a, b\}$. Note that X is only the closed set containing this $\{a, b\}$, that is, the set A, that is why $Cl(A) = X$. Now, if we are taking the set B and this set B is a singleton set $\{d\}$. What about $Cl(B)$? $Cl(B) = X \cap \{b, c, d\} \cap \{a, d\} \cap \{d\} = \{d\}$. From here, one thing is also clear that: this singleton set $\{d\}$ itself is the smallest closed set containing $\{d\}$.

If we are taking the well-known indiscrete topology, let us take X and take the topology $\mathcal{T}$, and this topology is indiscrete. If we are taking a nonempty subset A of X. The question is, what about the $Cl(A)$? Note that in the indiscrete topology, the closed sets are this X as well as the empty set. As A is non-empty set, so the intersection of all closed sets containing A, that will be X itself. Therefore, $Cl(A) = X$. But if we are changing the topology, let us take X with the discrete topology. In the case of discrete topology, what will happen? If we are taking any subset A of X, $Cl(A) = A$ because in the case of the discrete topology, every subset of A is closed. As A itself is the smallest closed set containing A. Therefore, $Cl(A) = A$.

Moving ahead, let us see a result. This result characterizes the concept of a closed set in terms of closure. If we are having a topological space $(X, \mathcal{T})$ and a subset A of X. Then A is closed if and only if $Cl(A) = A$. To prove this result, the first part is simple. For example, if we are assuming that $Cl(A) = A$, meaning A itself is a closed set because, by definition, $Cl(A)$ is a closed set. Moving ahead, if we are taking A as a closed set, then we have to justify that $Cl(A) = A$. Note that A is always a subset of A itself, and we have already seen that A is a subset of $Cl(A)$. Note that $Cl(A)$ is the smallest closed set containing A, and set theory says that A itself is the smallest. From here, we can conclude that both should be equal; that is, $Cl(A) = A$.

Moving to the next result on closure of sets, which states that $Cl(A) = A \cup A'$, that is if $(X, \mathcal{T})$ is a topological space and A is a subset of X, then $Cl(A) = A \cup A'$. Before proving this result, we will show that this $A \cup A'$ is a closed set. In order to prove this result, recall the concept of a closed set, that is characterization of a closed set given in terms of limit points. Then this is equivalent to showing that $(A \cup A')' \subseteq A \cup A'$. If we want to prove that this inclusion holds, it is equivalent to show that if x doesn't belong to $A \cup A'$, x does not belong to $(A \cup A')'$. So, begin with, x is not an element of $A \cup A'$, it means that x is neither an element of A nor an element of A'. Now, if x is not a limit point of A, it means that there exists an open set G containing this x such that $(G - \{x\}) \cap A$ is an empty set. But note that x is not an element of A. So, what can we write? This can also be written as $G \cap A = \emptyset$. Also, from here we can conclude one more thing, and that is, $G \cap A' = \emptyset$. The question is, how? The conclusion comes from here itself, because $G \cap A$ is the empty set. It means that for all $x \in G$, note that $(G - \{x\}) \cap A$ is the empty set. It means x cannot be a limit point of A, that is, $G \cap A' = \emptyset$. Combining these two results, we can say that $(G \cap A) \cup (G \cap A')$ is the empty set. It means that $G \cap (A \cup A')$ is empty, or $(G - \{x\}) \cap (A \cup A')$ is the empty set. From here, it is clear that x is not an element of the set of limit points of $A \cup A'$, that's the proof of this result. It justified that $A \cup A'$ is a closed set.

Moving ahead, coming to the main result, that is $Cl(A) = A \cup A'$. In order to justify this result, we have to justify two things. One is, $Cl(A) \subseteq A \cup A'$, and the second one is, $A \cup A' \subseteq Cl(A)$. Let us see one by one. For the first one, if we are looking at the definition of $Cl(A)$, that is, $Cl(A)$ is always a superset of A, and from the set theory, $A \cup A'$ will always be a superset of A. We have already seen that this is a closed set, but note that we have also seen that $Cl(A)$ is the smallest closed set containing A. So, from here, we can conclude that this $Cl(A)$ is a subset of $A \cup A'$. Moving ahead, that is coming to the second one. For the second one, note that A is always a subset of $Cl(A)$. What about if we are taking, or if we are talking about the set of limit points of A, that is A' will also be a subset of $(Cl(A))'$. Now, one thing we have already seen, $(Cl(A))'$ will always be a subset of $Cl(A)$. Because we have seen that a set P is closed if and only if $P' \subseteq P$. Because $Cl(A)$ is a closed set, that's why this will hold. Now, if we are clubbing these two, what are we getting? We are getting that $A \cup A' \subseteq Cl(A)$. So, this is the proof of the second one. First, we have already shown, therefore, $Cl(A) = A \cup A'$. This result may be used to compute the closure of a set in a simple way.

For example, if we are taking the real line, that is the set of real numbers with Euclidean topology. Let us talk about what is $Cl(\mathbb{Z})$, that will be nothing but $\mathbb{Z} \cup \mathbb{Z}'$. We have already seen that $\mathbb{Z}'$ is nothing but the empty set.

So, this $Cl(\mathbb{Z})$ will be $\mathbb{Z}$. Similarly, if we want to find out the closure of $\mathbb{Q}$, this will be nothing, but $\mathbb{Q} \cup \mathbb{Q}'$. We have already seen that this $\mathbb{Q}'$ is nothing but the set of real numbers, and therefore this $\bar{\mathbb{Q}}$ will always be equal to $\mathbb{R}$.

Let us take some more results. These are similar to the concept of limit points, what we have already shown. If we want to prove these three results, we will use the concept, that is $Cl(A) = A \cup A'$ plus the consequences or the results we have discussed in the case of limit points of a set. Beginning with the first one, i.e., $A \subseteq B \Rightarrow Cl(A) \subseteq Cl(B)$. If we compute $Cl(A)$, by the previous result, this is nothing but $A \cup A'$. This can be written as a subset of B$\cup B'$, because A is a subset of B, and this is nothing but $Cl(B)$. Therefore, $Cl(A) \subseteq Cl(B)$.

Moving to the next one, if we want to justify the result of closure for the union, i.e., $Cl(A \cup B) = Cl(A) \cup Cl(B)$. Then $Cl(A \cup B)$ can be written as $(A \cup B) \cup (A \cup B)'$, or this can be written as $(A \cup B) \cup (A' \cup B')$. The result which we have shown in the case of the limit points and by using commutativity of sets, this can be written as $(A \cup A') \cup (B \cup B')$, and this is nothing but $Cl(A) \cup Cl(B)$. If we are coming to this third one, i.e., $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$, it's simple and follows from the first one itself because $A \cap B \subseteq A$. Also, $A \cap B \subseteq B$. If we are using the first one, then $Cl(A \cap B) \subseteq Cl(A)$, and $Cl(A \cap B) \subseteq Cl(B)$. From here, we can conclude that $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$. It is to be noted that the equality may not hold in general. For example, if we are taking real line with Euclidean topology. Let us take a set A, that is an open interval $(2, 3)$, and B is an open interval $(3, 4)$. If we are computing $Cl(A)$, that will be this closed interval $[2, 3]$, and $Cl(B)$ will be the closed interval $[3, 4]$. What about this $Cl(A) \cap Cl(B)$? Note that this will be a single set $\{3\}$. If you are computing $A \cap B$, that will be nothing but the empty set. Therefore, $Cl(A \cap B)$ will be the empty set. Thus, it shows that equality may not hold in general.

Moving to the next concept, that is the concept of dense sets. This concept is defined in terms of the closure of a set. The definition is, Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then A is called dense if $Cl(A) = X$. For example, if we are taking a topological space, let us take $X = \{a, b, c\}$. Put a topology $\mathcal{T}$ on it, let us take this $\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. If we see the closed sets in this topology, the closed sets are: $X, \emptyset, \{b, c\}, \{a, c\}, \{c\}$. Now, if we are taking a set, for example, A as $\{b, c\}$, and we are trying to find out $Cl(A)$, that will be $X \cap \{b, c\}$, that is, A itself. Note that this is not equal to X, meaning is, this A is not dense. Now, if we are changing and we are taking a set B, that set is nothing but simply $\{a, b\}$. If we are trying to find out $Cl(B)$, try to find out the intersection of all closed sets containing B. Note that it is X, and therefore, B is dense.

Let us take some more general examples. Why not let us take the real line with Euclidean topology. We have already seen that $Cl(\mathbb{Q}) = \mathbb{R}$. It means that the set of rational numbers is dense, but we have also seen that if we are taking $Cl(\mathbb{Z})$, that will be $\mathbb{Z}$ itself, which is not equal to $\mathbb{R}$. Therefore, the set of integers is not dense. Coming to the well-known discrete topology. Let us take a topological space $(X, \mathcal{T})$, and we are taking the topology as the discrete topology. We already know that $Cl(A) = A$, for all subsets A of X. So, what will happen in this case? $Cl(X) = X$, and if we are taking a proper subset of X, note that its closure cannot equal X. So, what will happen in the case of discrete topological spaces? X is the only dense set. Moving to the next example, let us take $(X, \mathcal{T})$, where this is an indiscrete topology. What have we seen in the case of an indiscrete topology? if we are taking any subset of X, provided this is non-empty, what about $Cl(A)$? It is always equal to X. Meaning, in the case of indiscrete spaces, all non-empty subsets are dense.

Moving ahead, what we have seen, we have defined dense sets by using the concept of closure of a set. This is a characterization of a dense set that is given in terms of an open set. What does this result state? This result says that if we are having a topological space $(X, \mathcal{T})$, and a subset A of X, A is dense iff $\forall G \in \mathcal{T}, G \neq \emptyset, G \cap A \neq \emptyset$. Let us prove the result in two parts. So, first, we are assuming that A is dense. Let us try to justify that for all non-empty open sets G, $G \cap A$ will always be the empty set. We are going by contradiction. If possible, let us take this $G \cap A$ is empty. Now, if we are taking an element x, which is from A^c, and take an open set G, which is containing this x. So, we are taking an open set G such that $x \in G$. Now, what will happen? Because $G \cap A$ is always an empty set, from here, we can conclude that this x is not a limit point of A. So, what is happening for all $x \in A^c$? x is not a limit point of A, it means that A^c is not a subset of A'. Now, if we are looking at what is the closure, closure of a set A is given by $A \cup A'$ and note that X can be written as $A \cup A^c$. Also, what we are justifying that the elements of A^c are not limit points of A, meaning is $Cl(A)$ cannot be equal to X, which is contradicting the assumption that A is dense.

Coming to the converse part of this result. Let us assume that for all non-empty open sets G, $G \cap A$ is non-empty. So, we are assuming it, and let us justify that A is dense. Note that if $A = X$, then obviously, $Cl(A) = X$. But what will happen if A is not equal to X. If A is not equal to X, obviously, A^c cannot be an empty set, and if this is not an empty set, let us take an element x of A^c. Now, note that it is already given that this $G \cap A$ is always non-empty, G is an open set, and obviously, this is a non-empty open set. So, what we have shown here is that x is an element of A', that is x is a limit point of A. It means that $A^c \subseteq A'$. Now, if we are looking for X, that is the union of A and A^c, note that this is a subset of $A \cup A'$, and that is equal to the $Cl(A)$. So, finally, from here, we can conclude that $Cl(A) = X$, or A is a dense set.

These are the references.

That's all from this lecture. Thank you.