Course Name: Essentials of Topology
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Welcome to Lecture 17 on Essentials of Topology.

In this lecture, we will discuss the concept of limit points of sets. We will see how a closed set can be characterized by using the concept of limit points. If we are having a topological space $(X, \mathcal{T})$, an element $x \in X$, and a subset A of X. Then x is a limit point of the set A, if every open set G containing x, G contains at least one element of A different from x. Mathematically, if G is any arbitrary open set containing x, then $(G - \{x\}) \cap A \neq \emptyset$. We will use this notation A' to denote the set of all the limit points of A.

Let us take some of the examples. If we are taking $X = \{a, b, c, d\}$, let us put a topology on it, that is, $\mathcal{T} = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. Now, if we are taking a set A, that is a subset of X, and this set is, for example, $\{b, c\}$, the question is, what about A', that is, the set of limit points of A? What we have to do is that we have to check for all the points or all the elements of X. For example, if we are taking $x = a$, then the open sets containing a are either this X or this set $\{a, b\}$. Now, $(X - \{a\}) \cap A \neq \emptyset$ and $(\{a, b\} - \{a\}) \cap A \neq \emptyset$. So, one can say that a is a limit point of A. Similarly, if we are taking another point, for example, $x = b$, what will happen here? There are two open sets containing b, one open set is X, and the second is $\{a, b\}$ itself. But in the case of this $\{a, b\}$, $(\{a, b\} - \{b\}) \cap A = \emptyset$. Therefore, b cannot be a limit point of A. Moving ahead, let us take $x = c$, then the open sets containing c are X as well as $\{c, d\}$. Let us see in the case of the second one, that is, if we are finding out $(\{c, d\} - \{c\}) \cap A$, note that this is again an empty set, therefore c is not a limit point of A. But if we are looking for d, that is if we are taking $x = d$, then the open sets containing d are again X as well as $\{c, d\}$. Now, $(\{c, d\} - \{d\}) \cap A \neq \emptyset$ and $(X - \{d\}) \cap A \neq \emptyset$. Therefore, d is a limit point of A. Finally, the set of limit points of A is $\{a, d\}$.

Let us take some more examples, the well-known examples from topology, that is, our discrete topological spaces and indiscrete topological spaces. In the case of discrete topological spaces, if we are taking any subset of X, what will happen? A' will always be an empty set. Why? The answer is, because we know that for all $x \in X$, singleton set $\{x\}$ is a member of the discrete topology, and now what will happen? If we are choosing G like this $\{x\}$, then $(G - \{x\}) \cap A = \emptyset$, meaning is, any element of X cannot be a limit point of A, and that is why A' is an empty set. But if we are coming for indiscrete topological spaces. Here, we are taking a subset of X having at least two elements, what will happen? We know that in the case of the indiscrete topological spaces, open sets are either X or an empty set. So what are we taking? If we are taking any $x \in X$, as the open set containing x is only this X itself, and $(X - \{x\}) \cap A \neq \emptyset$, therefore, every x will be an element of A' or $A' = X$. What about if the number of elements in A becomes one? Need to think. This is a simple one.

Moving ahead, let us take Euclidean topology on the set of real numbers and a set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Our motive is to show that 0 is a limit point of this set. What we have to justify is that let us take any open set G which is containing 0, and if 0 is an element of G, then as per our definition, there exists an open interval (a, b) so that $0 \in (a, b) \subseteq G$, a and b are real numbers, and $a < b$. Now, the question is, whether this open interval will contain some elements of A. The answer is yes, because of the Archimedean property. We know that for all $\epsilon > 0$, there exists this $n \in N$ such that $1/n < \epsilon$. So what will happen? The interval (a, b) contains an element of this set A and not only one element. This will contain an infinite number of elements of this set A. So, finally, what will happen? This $(a, b) \cap A$ will not be an empty set. Therefore, this $G \cap A$ is not an empty set, or one can write that $(G - \{x\}) \cap A \neq \emptyset$. Therefore, 0 is the limit point of set A, which consists of the real numbers of this particular form.

Coming to the Euclidean topology, let us take intervals, that is, either we are taking an open interval $(2, 5)$ or semi-open interval $(2, 5]$, or we are taking a closed interval $[2, 5]$. How the set of limit points of these sets look like? The answer is here. If we are taking any element x inside this $(2, 5)$, one can construct an open set (open interval) G containing this element x so that $(G - \{x\}) \cap A$ will contain an infinite number of elements of interval $(2, 5)$. So, all the elements lying in $(2, 5)$ will be the limit points of all of these sets. Coming to the end points. If we are taking $x = 2$, again, we can construct open intervals and note that these open intervals will contain an infinite number of elements of interval $(2, 5)$, and the same case will be if we are considering $x = 5$. So, what will happen? All the elements of this interval $[2, 5]$, are limit points of this set. But if we are taking an element or a real number outside this $[2, 5]$, what we can do that we can construct some open sets which don't intersect with the intervals, and therefore, such points cannot be a limit point, and this is the justification about this example.

Similarly, the set of limit points of the set of rational numbers is $\mathbb{R}$. How is it possible? If we are taking any real number x and we are taking any open set G containing this x, what will happen? Again, there exists an open interval (a, b) such that $x \in (a, b) \subseteq G$, a and b are real numbers, $a < b$. Now, what will happen if we want to visualize the intersection of this open interval (a, b) with the set of rationals? Note that this will be an infinite set because this open interval (a, b) contains an infinite number of rational numbers. So, what will happen? Finally, $(G - \{x\}) \cap \mathbb{Q}$ will always be nonempty, and therefore, this $\mathbb{Q}'$ will always be equal to $\mathbb{R}$. But if we are taking the set as set of integers. In this case, the set of limit points will always be an empty set. Meaning is, no element of $\mathbb{R}$ can be a limit point of this set. Why? Again, the answer is simple because integers cannot be the limit point of this set. Why? As we can construct an open set which is consisting an integer only, i.e., $(G - \{x\}) \cap \mathbb{Z}$ becomes an empty set, and that integer cannot be a limit point. Even if we are taking any other real number, we can again construct an open set containing the real number such that the intersection with $\mathbb{Z}$ is an empty set.

Let us characterize the closed sets by using the concept of limit points of a set. The set A is closed if and only if it contains all of its limit points, that is A' is a subset of A. Let us prove this result. We are assuming that A is closed, and let us try to justify that this A' is a subset of A. If you want to justify this result, this is equivalent to just show that if x is not an element of A, x cannot be an element of A'. Let us show it. If x doesn't belong to A, what does it mean? Meaning is, $x \in A^c$, but what is given to us, that is, A is closed. Therefore, A^c is open. It means that A^c is an open set, and this open set is containing x. Now, as we know that $A \cap A^c = \emptyset$, or we can also write something like $(A - \{x\}) \cap A^c = \emptyset$. It means that x is not a limit point of A, that's all about this result.

Moving ahead, let us assume that this A' is a subset of A and try to justify that A is closed. It is equivalent to show that A^c is open. Now, if we want to show that A^c is open, let us take $x \in A^c$. It means that x cannot be an element of A. What is given to us? If x is not an element of A, x cannot be an element of A'. Now, coming to the definition of a limit point. If x is not an element of A', it means that there exists an open set G_x which contains x, with the feature that $(G_x - \{x\}) \cap A = \emptyset$. But note that x is not an element of A. So, from here, we can also write that $G_x \cap A = \emptyset$, or we can write that $G_x \subseteq A^c$. From here, we can deduce that this A^c can be expressed as a union of such G_x, where $x \in A^c$. If this is the expression, then from here, we can conclude that A^c is open. Because A^c is an arbitrary union of open sets, and we know that an arbitrary union of open sets is open.

Let us take some examples based on this result. For example, what we have already seen that if we are taking real line with Euclidean topology, we have already seen that if we are taking this open interval $(2, 3)$, then the set of its limit points was this $[2, 3]$. Note that this closed interval $[2, 3]$ is not contained in this open interval $(2, 3)$. Therefore, this open interval $(2, 3)$ is not closed. But if we are taking the set of integers, we have shown that $\mathbb{Z}'$ is an empty set, and note that in this case, this empty set is a subset of $\mathbb{Z}$. Therefore, $\mathbb{Z}$ is a closed set. Even if we are looking the example of set of rationals, what we have seen that it's set of limit points is $\mathbb{R}$. But note that this $\mathbb{R}$ is not a subset of $\mathbb{Q}$, therefore this set $\mathbb{Q}$ is not a closed set. So, these are some examples, where we see that how to characterize the closed sets by using the result that A is closed if and only if $A' \subseteq A$.

Moving ahead, let us discuss some results related to limit points. The first one is: if $A \subseteq B$, then $A' \subseteq B'$. This is simple to deduce. For example, if we are taking any $x \in A'$, it means that x is a limit point of A, meaning is, for all open sets G containing x, $(G - \{x\}) \cap A$ is nonempty, or we can also write that $(G - \{x\}) \cap B$ is nonempty, because it is given to us that $A \subseteq B$, or we can say that x is an element of B'.

Moving ahead, let us see the second one, that is, $(A \cup B)' = A' \cup B'$. We have to justify results in two parts. The first one is: we have to show that $(A \cup B)' \subseteq A' \cup B'$, and the second one, we have to justify that $A' \cup B' \subseteq (A \cup B)'$. Note that this second one is trivial because we know that $A \subseteq A \cup B$ and

$B \subseteq A \cup B$. From here, we can conclude that A' and B', both are subsets of $(A \cup B)'$ or $A' \cup B' \subseteq (A \cup B)'$.

Coming to the first one, let us take an element $x \in (A \cup B)'$, that is, x is a limit point of $A \cup B$. Then what will happen? Meaning is, for all open sets G containing x, $(G - \{x\}) \cap (A \cup B) \neq \emptyset$, or what we can say that $((G - \{x\}) \cap A) \cup ((G - \{x\}) \cap B)$ is nonempty. Use the set theory, if union of two sets is nonempty, it means that at least one set is nonempty, that is, $(G - \{x\}) \cap A$ is nonempty or $(G - \{x\}) \cap B$ is nonempty. It means that $x \in A'$ or $x \in B'$, or we can conclude from here that $x \in A' \cup B'$. That's the proof for this first part. Therefore, the second holds, that is, $(A \cup B)' \subseteq A' \cup B'$.

Coming to the third one which is simple, and it follows from the first part because $A \cap B$ is a subset of A, and $A \cap B$ is also a subset of B. If we are using the first result, then we can say that $(A \cap B)'$ is a subset of A' and $(A \cap B)'$ is also a subset of B'. Combining both, we can say that $(A \cap B)' \subseteq A' \cap B'$. But note that, the equality may not hold in this case. We can construct some counter-examples for it. For example, let us take the set of real numbers with Euclidean topology. If we are taking a set A, that is an open interval $(2, 3)$, and we are taking that B is an open interval $(3, 4)$, what about $A \cap B$? $A \cap B$ is nothing but an empty set, and what about the set of limit points of this set, that will be the empty set. But if we are looking for A', A' is nothing but a closed interval $[2, 3]$. What about B'? That is given by closed interval $[3, 4]$. Now, if we are looking for $A' \cap B'$, that is nothing but a singleton set $\{3\}$. Note that both are not equal.

These are the references.

That's all from this lecture. Thank you.