Course Name: Essentials of Topology
Professor Name: S.P. Tiwari
Department Name: Mathematics & Computing
Institute Name: Indian Institute of Technology(ISM), Dhanbad
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Welcome to Lecture 16 on Essentials of Topology.

In this lecture, we will study the concept of subbases and neighborhoods. In the previous lecture, we have seen the relationship between open sets and bases. Now, we will see how these two concepts are related to the concept of open sets. Begin with the concept of subbases. If we recall the relationship between bases and a topology, we have seen that for a given non-empty set X, if we are taking $\mathcal{B}$ as a subset of the power set of X, then $\mathcal{B}$ generates a topology on X, provided it satisfies two conditions. The first one was: X should be expressed as a union of members of this $\mathcal{B}$, and the second condition was regarding the intersection of two members of this $\mathcal{B}$, that is, for all $B_1, B_2 \in \mathcal{B}$ and for all $x \in B_1 \cap B_2$, there exists some $B_3 \in \mathcal{B}$ with the condition that $x \in B_3 \subseteq B_1 \cap B_2$. What our motive is and what is the motivation behind this concept of subbases? Actually, we are looking for a smaller structure than this $\mathcal{B}$. So, if we are beginning from here, can we still generate some topology on X? This is the question. The thing is that here are two conditions, can we relax the conditions too? The answer is yes.

Begin with some $\mathcal{B}'$, that is again a subset of the power set of X, and use only the condition so that the union of members of B' is X, then we can again generate a topology $\mathcal{T}$ on X, and if this is possible, we call this $\mathcal{B}'$ a subbasis. The question is, how is it possible? How to do it? The answer is simple. Begin with, let us take an example. Let us take a set $X = \{a, b, c\}$. Let us take some $\mathcal{B}' = \{\{a, b\}, \{b, c\}\}$. Now, if we are taking intersection of these two, what can we get? This $\{a, b\} \cap \{b, c\} = \{b\}$. We can see that this $\mathcal{B} = \{\{a, b\}, \{b\}, \{b, c\}\}$ will be a basis because one can see that their union is X. If we are looking for the second property that is also satisfied by this $\mathcal{B}$ and by this $\mathcal{B}$, we can put a topology on X, that is $\{\emptyset, X, \{a, b\}, \{b\}, \{b, c\}\}$. So, we have begun with this $\mathcal{B}'$, we are coming to $\mathcal{B}$, and from $\mathcal{B}$, we can put a topology. What is the relation between $\mathcal{B}'$ and $\mathcal{B}$? The answer is that the members of $\mathcal{B}$ are (finite) intersections of members of $\mathcal{B}'$. This is the key idea from where one can begin to generate a topology on a set X.

So, formally, a subbasis $\mathcal{B}'$ for a topology $\mathcal{T}$ on X is a collection of subsets of X, whose union equals X. The question is, how can we find the topology in a general way? The answer is, if we are taking some $\mathcal{B}$, which is nothing but a collection of the finite intersection of members of $\mathcal{B}'$, then we can show that this $\mathcal{B}$ is a basis, and if $\mathcal{B}$ is a basis, this will always generate a topology $\mathcal{T}$ on X. In order to show that this $\mathcal{B}$ is a basis, begin with the first one, which is, if we are taking any element of X, then what will happen? There exists a member B' in this $\mathcal{B}'$ and note that $\mathcal{B}'$ is subset of this $\mathcal{B}$, because we have taken this $\mathcal{B}$ as a collection of finite intersection of members of $\mathcal{B}'$. Meaning is, there exists some $B' \in \mathcal{B}$ such that $x \in B' \subseteq X$, or X can be expressed as the union of members of $\mathcal{B}$. Moving ahead, let us take two elements of this $\mathcal{B}$, B_1 and B_2. So, what about B_1 and B_2? B_1 will be some finite intersection of members of $\mathcal{B}'$. Therefore, this B_1 can be expressed as something like $H_1 \cap H_2 \cap ... \cap H_n$, where $H_1, H_2, ..., H_n$ are members of $\mathcal{B}'$. Similarly, this B_2 can be written as something like $K_1 \cap K_2 \cap ... \cap K_m$, and note that these $K_1, K_2, ..., K_m$, are also in $\mathcal{B}'$. So, what about $B_1 \cap B_2$? Meaning is, because B_1 is a finite intersection of members of $\mathcal{B}'$, and B_2 is also a finite intersection of members of $\mathcal{B}'$. So, we can write $B_1 \cap B_2$ as a finite intersection of members of $\mathcal{B}'$, and if so, $B_1 \cap B_2$ is an element of $\mathcal{B}$. Thus, if we are taking any $x \in B_1 \cap B_2$, there exists B_3, and that B_3 is nothing but $B_1 \cap B_2$ itself in this $\mathcal{B}$ satisfying the criteria that $x \in B_3 \subseteq B_1 \cap B_2$. So, we can conclude that the $\mathcal{B}$, which we have taken as a collection of the finite intersection of members of $\mathcal{B}'$, is a basis for some topology on X.

Let us take some of the examples. For example, if we are taking $\mathcal{B}' = \{(a, \infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\}$. The question is, is it a subbasis for the well-known Euclidean topology on $\mathbb{R}$? The answer is yes. First one, if we are looking for what is $\mathcal{B}'$, the members are subsets of $\mathbb{R}$. Second thing, if we are looking for (a, ∞) as well as $(-\infty, b)$, we have already seen that such intervals are members of the Euclidean topology. If we are taking the real numbers a and b such that $a < b$, and if we are taking the intersection of intervals (a, ∞) as well as $(-\infty, b)$. What will be this intersection? Actually, this intersection will be nothing but (a, b). We have already seen that a collection of open intervals (a, b), a and b are real numbers, $a < b$, this is nothing but the basis for Euclidean topology on $\mathbb{R}$. It is to be noted here that $\mathbb{R}$ can also be expressed as a union of such intervals.

Moving ahead, let us take one more example. If we are taking $\mathcal{B}' = \{[a,b] : a,b \in \mathbb{R}, a < b\}$, this is a subbasis for the discrete topology on $\mathbb{R}$, because $\mathbb{R}$ can be expressed as the union of such intervals, as for all $x \in \mathbb{R}$, note that we can construct a closed interval $[x-1, x+1]$, which is a subset of $\mathbb{R}$. So, the union of such members can be $\mathbb{R}$. Moving ahead, if we are taking any real number x, that is for all $x \in \mathbb{R}$, what we can do that if we are taking the intersection of such intervals, that is $[x-1, x] \cap [x, x+1]$, note that $x-1$ is less than x and x is less than $x+1$. So, what exactly will we get? It is nothing but the singleton set $\{x\}$. Thus, what we have concluded that for all $x \in \mathbb{R}$, singleton set $\{x\}$ is a member of $\mathcal{B}$, and note that the collection of all singleton sets forms a basis for discrete topology.

Moving ahead, let us take one more example, that is the example of co-finite topology on the set of real numbers. The question is if we are taking $\mathcal{B}$ as sets that are the complement of single point sets; such collection forms a basis? This $\mathcal{B}$ is something like a collection of $\mathbb{R} - \{x\}$, and this x is a real number. So, whether this is a basis for co-finite topology. Let us see. For example, if we are taking the set $\mathbb{R} - \{0, 1\}$. Obviously, note that this will always belong to co-finite topology because its complement is finite. So, this is an open set G, and this open set G contains an element; let us take that element as 2. The question is, can we find some $B \in \mathcal{B}$ such that $2 \in B \subseteq G$. The answer is no. We cannot do it because if we are taking B something like $R - \{0\}$, note that this will contain 1, and that cannot be a subset of G. Even, if we are taking this B as $R - \{1\}$, that will also contain 0, and therefore this B cannot be a subset of G. Also, if we are taking any other real number r, this is $\mathbb{R} - \{r\}$, and r is neither equal to 0 nor equal to 1. Then this B contains both 0 and 1, and therefore this B cannot be a subset of G. Therefore, such collection, that is $\mathcal{B}$ consisting of a complement of singleton sets in $\mathbb{R}$, cannot be a basis for the co-finite topology on $\mathbb{R}$. Whether by using this one, can we generate a topology, when we are assuming this $\mathcal{B}$ as a subbasis. So, if we are writing, instead of $\mathcal{B}$, let us take this is $\mathcal{B}' = \{R - \{x\} : x \in \mathbb{R}\}$. Note that, we can show that this is a subbasis for the co-finite topology on $\mathbb{R}$. The question is how? Can we justify it? The answer is, if we are taking the union of such members of $\mathcal{B}'$, note that, that will always be equal to $\mathbb{R}$. So, this $\mathcal{B}'$ is a subbasis for a topology and that is co-finite topology because if we are looking for the sets of this form, that is $\mathbb{R} - \{x_1, x_2, ..., x_n\}$. Then this set can always be expressed as $(R - \{x_1\}) \cap (R - \{x_2\}) \cap ... \cap (R - \{x_n\})$ and collection of sets of the form $\mathbb{R} - \{x_1, x_2, ..., x_n\}$ forms a basis for co-finite topology on $\mathbb{R}$. So, what we have seen that such collection cannot be a basis but it can turn out to be a subbasis for co-finite topology on $\mathbb{R}$.

Moving ahead, let us conclude the relationship between subbasis, basis and topology. What we have seen that if we are having a topology $\mathcal{T}$, the basis $\mathcal{B}$ is a subcollection of $\mathcal{T}$. Also, note that the subbasis $\mathcal{B}'$ is a subset of this $\mathcal{B}$. So, this is the relationship between subbasis, basis as well as topology. One more point to be noted here. We have already seen that every topology is a basis, but a basis may not be a topology. The same relationship exists between a subbasis and a basis. Every basis itself is a subbasis but a subbasis may not be a basis, which we have seen in the case of the co-finite topology. Finally, if we are beginning with a subbasis, how the open sets are related to the members of the subbasis? The answer is here. If we are taking an open set, that is a member of topology. So, this open set can be expressed as a union of the finite intersection of members of the subbasis. So, begin with a subbasis, take the finite union of members, and take their intersection; one can get an open set. Similarly, if we are looking for the concept of basis, take members of basis, take their union, and that we are getting the concept of an open set. Finally, if we are looking for topology, we know that members of $\mathcal{T}$ are open sets.

Moving to the next one, that is the concept of a neighborhood. Let us see the definition first. Let $(X, \mathcal{T})$ be a topological space and $x \in X$. Then $N \subseteq X$ is called a neighborhood (nbd) of x in X if there exists an open set G such that $x \in G \subseteq N$. Roughly speaking, what does it mean that if we are having a set X and if we are taking a subset of it, for example, this is our N. Then whenever we want to find out the neighborhood of an element, obviously from the definition, it can be seen that x will always be element of N, that is why we are taking x like this one, that is x is inside N. We say that this N is a neighborhood of x, if we can find an open set G such that this G is containing x and that lies inside N, that is, $x \in G \subseteq N$.

Let us take some of the examples. Begin with the real line with Euclidean topology on it. We are taking N as, for example, this is the closed interval $[0, 1]$. Let us take an element x, that is $1/2$. The question is, whether this N

is a neighborhood of $1/2$. The answer is yes, because this $1/2$ belongs to an open set, and that open set is nothing but the open interval $(0, 1)$, that is a subset of this N. Therefore, we can say that N is a neighborhood of $1/2$. If we are taking x as $1/4$, that is again in $\mathbb{R}$, whether N is a neighborhood of $1/4$? The answer is yes, because $1/4$ belongs to an open interval, let us take this as $(0, 1/2)$, that is open set in the Euclidean topology, and that will always be a subset of N. But if we are taking 0 itself, that is $x = 0$, the question is, whether N is a neighborhood of 0. The answer is no. Why? Obviously this is in N, can we construct an open set G here which is containing this 0, and that is a subset of N. The answer is no. Such types of constructions we have already seen when we have compared this Euclidean topology and lower limit topology, that is why, we are not doing the same calculation again. Thus, we cannot construct any open set G, which contains 0 and is a subset of this N. But, if we are changing the topology, that is, instead of taking this Euclidean topology, if we are taking the lower limit topology. Then we can see that this N is a neighborhood of 0 because this 0 belongs to this open set $[0, 1)$, which is a subset of N. Note that this semi-open interval $[0, 1)$ is a member of the lower limit topology.

Moving ahead, what we can see or what we can observe from the definition of the neighborhood is that if we are taking any element x of X and the subset N of X, then we say that N is a neighborhood if $x \in G \subseteq N$. So, if we are thinking in another way, that is if we are having an open set G and this open set is containing x, and if we are taking the supersets of G, for example, N_1 is a superset of G, N_2 is another superset of G, let us take another superset N_3 of G. So, what will happen? Note that $x \in G \subseteq N_1$, $x \in G \subseteq N_2$, and $x \in G \subseteq N_3$. Meaning is, the supersets of this open set G satisfying the criteria what the definition of neighborhood require. Thus, if we are having a set or we are having an open set containing an element and if we are taking the supersets, supersets will always be neighborhoods of that particular element.

Let us see one more concept about open sets itself, that is we can always write $x \in G \subseteq G$. Note that this can be written for all $x \in G$. From here, we can conclude that every open set is a neighborhood of each of its points or elements. Even we can characterize a general result here, that if we are having a topological space $(X, \mathcal{T})$, then we can say that this N is open if and only if it is neighborhood of each of its points or each of its elements. The result is simple. For example, if N is open, then what we can write that for all $x \in N$, we can write $x \in N \subseteq N$. Meaning is, N is a neighborhood of x, that is N is a neighborhood of each of its points. Moving ahead, if we are assuming that N is a neighborhood of each of its elements, then what will happen? Meaning is, for all $x \in N$, there exists an open set G_x such that $x \in G_x \subseteq N$. We can observe that this N is nothing but a union of G_x, $x \in N$. These types of results we have already seen earlier, and if G_x is an open set, an arbitrary union of open sets is open. Therefore, N is open. That's the proof of this simple result.

Moving ahead, let us discuss some more results and conclusions about neighborhoods. Let $(X, \mathcal{T})$ be a topological space, $x \in X$ and N_x be collection of all nbds of x. Then this collection has some important features. What are the features? The first one is that this N_x cannot be an empty set. What does it mean? It means that for every element of X there will exist at least one neighborhood. Note that, one neighborhood will be X itself because X is always an open set, and for all x belongs to X, we can write $x \in X \subseteq X$. Therefore N_x cannot be empty. The second thing is simple one, which we have already observed that $N \in N_x \Rightarrow x \in N$. It follows from the definition of the neighborhood itself because the definition says that $x \in G \subseteq N$, where G is an open set. Moving ahead, let us show that $N \in N_x, N \subseteq M \Rightarrow M \in N_x$. Meaning is, a superset of a neighborhood is a neighborhood. The justification is simple, because if $N \in N_x$, it means that there exists an open set G such that $x \in G \subseteq N$. Note that, as N is a subset of M, we can write that $x \in G \subseteq M$, or we can conclude that $M \in N_x$. Moving ahead, if we are taking two elements in N_x, that is $M, N \in N_x$, it means that M and N are neighborhoods of x. What about their intersection? Whether $M \cap N$ will also be in N_x? The answer is yes. How is it possible? Let us take M as a neighborhood of x. So, what will happen? There exists, or if we are writing $M \in N_x$, meaning is, there exists an open set G_1 such that $x \in G_1 \subseteq M$. Similarly, if N is an element of N_x, again there exists an open set G_2 such that $x \in G_2 \subseteq N$. If we are clubbing these two, what we can conclude that $x \in G_1 \cap G_2 \subseteq M \cap N$. Also, we already have seen that if G_1 and G_2 are open, that is, if G_1 and G_2 are members of topology, then their intersection too. Therefore, $M \cap N$ is a neighborhood of x, or $M \cap N \in N_x$. What about the union? So, one thing is simple from here: $M \cup N$ will always be a member of N_x, that is, $M \cup N$ will always be a neighborhood of x, if M and N are neighborhoods of x. Why?

Because $M \cup N$ is a superset of M, and the superset of a neighborhood is a neighborhood.

These are the references.

That's all from this lecture. Thank you.