Course Name: Essentials of Topology
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Welcome to Lecture 15 on Essentials of Topology.

We have already seen a number of topologies on the set of real numbers. In this lecture, we will study a topology on $\mathbb{R}^2$, namely Euclidean topology. For the study of this topology, we will use the concepts from the basis for topology as well as the concept of metric spaces. Begin with, let us take

$$\mathcal{B} = \{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}, a < b, c < d\}.$$

Note that, $(a, b) \times (c, d)$ is representing an open rectangle bounded by four lines $x = a, x = b, y = c$ and $y = d$. Let us see that this is a basis. For which, we have to justify that $\mathcal{B}$ satisfies two conditions.

Begin with the first one, where we have to justify that $\mathbb{R}^2$ can be expressed as the union of members of $\mathcal{B}$. For it, let us take any ordered pair $(x, y) \in \mathbb{R}^2$. Then, we can find a $B \in \mathcal{B}$ so that $(x, y) \in B \subseteq \mathbb{R}^2$, and this B can be constructed in a simple way. The idea is, if we are taking $(x, y) \in \mathbb{R}^2$, let us take $B = (x - 1, x + 1) \times (y - 1, y + 1)$. So, corresponding to every element $(x, y) \in \mathbb{R}^2$, we can find a $B \in \mathcal{B}$ satisfying $(x, y) \in B \subseteq \mathbb{R}^2$, and therefore $\mathbb{R}^2$ can be expressed as a union of members of $\mathcal{B}$.

Moving to the second condition, which is required to show that $\mathcal{B}$ is a basis. Let us take $B_1, B_2 \in \mathcal{B}$, where $B_1 = (a_1, b_1) \times (c_1, d_1)$ and $B_2 = (a_2, b_2) \times (c_2, d_2)$. Also, let us take an element $(x, y) \in B_1 \cap B_2$, which is precisely $((a_1, b_1) \times (c_1, d_1)) \cap ((a_2, b_2) \times (c_2, d_2))$. Then $(x, y) \in (a_1, b_1) \times (c_1, d_1)$ and $(x, y) \in (a_2, b_2) \times (c_2, d_2)$. From here, it is clear that $a_1 < x < b_1, a_2 < x < b_2, c_1 < y < d_1$ and $c_2 < y < d_2$. Now, let us take $a = max\{a_1, a_2\}$, $b = min\{b_1, b_2\}$, $c = max\{c_1, c_2\}$ and $d = min\{d_1, d_2\}$. Then we can find a $B_3 \in \mathcal{B}$ which is given as $(a, b) \times (c, d)$ such that $(x, y) \in B_3 \subseteq B_1 \cap B_2$. Therefore, $\mathcal{B}$ is a basis for some topology on $\mathbb{R}^2$.

What will be that topology? We will see it and come up with the structure of that topology. Before going to see the structure of the topology, let us move to the concept of metric topology. We have already seen that for a metric space (X, d), we can find a topology, that is, the metric topology given by

$$\mathcal{T}_d = \{G \subseteq X : \forall x \in G, \exists r > 0 \text{ such that } B(x, r) \subseteq G\}.$$

From the definition of basis and the definition of topology, it is clear that $\mathcal{B} = \{B(x, r) : x \in X, r > 0\}$ is a basis for metric topology.

Let us take an example. Here, we are taking X as the set of real numbers and $d(x, y) = |x - y|, x, y \in \mathbb{R}$. Then, we have already seen that in this case, $B(x, r) = (x - r, x + r), r > 0$. One thing is interesting here: If we are taking any open interval (a, b), where a and b are real numbers with $a < b$, we can see that every open interval (a, b) can be visualized as an open interval $(x - r, x + r)$. How? Let us take $x = (a + b)/2$ and $r = (b - a)/2$. Then $x - r = a$ and $x + r = b$. So, finally, we have shown that every open interval (a, b) can be visualized as an open interval of the form $(x - r, x + r)$. Thus, what we can conclude is that if on the set of real numbers, we are taking Euclidean metric for topology, one can get the Euclidean topology, given by $\mathcal{T}_e = \{G \subseteq \mathbb{R} : \forall x \in G, \exists a, b \in \mathbb{R} \text{ with } a < b \text{ such that } x \in (a, b) \subseteq G\}$.

Let us see some more examples regarding the metric topology. We have already seen that the metric given by $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ is the Euclidean metric on $\mathbb{R}^2$, where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. We have also seen that the open ball is given by $B(x, r) = \{y \in \mathbb{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < r\}$. Let us denote the topology induced by this metric by $\mathcal{T}_d$, and with this open ball, we can construct a basis $\mathcal{B} = \{B(x, r) : x \in \mathbb{R}^2, r > 0\}$ for it. Also, it is clear that whenever we are looking for an open ball, that looks like an open circle in the case of metric d on $\mathbb{R}^2$ defined here.

Moving ahead, let us take one more metric that we have studied. This is the example of the taxicab metric defined by $d_T(x, y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. If we are looking for an open ball here centered at x and radius r, this is given by $B(x, r) = \{y \in \mathbb{R}^2 : |x_1 - y_1| + |x_2 - y_2| < r\}$. Let us denote the topology induced by this metric by $\mathcal{T}_{d_T}$, and there will be a basis for it, that is, $\mathcal{B} = \{B(x, r) : x \in \mathbb{R}^2, r > 0\}$. We have also seen that whenever we are discussing this open ball, these open balls look like a diamond type of structure.

Moving ahead, let us summarize the concepts of bases that we have studied and club them together. What we have seen is that the collection of open rectangles forms a basis for some topology on $\mathbb{R}^2$. Let us denote this topology by $\mathcal{T}$. Similarly, we have seen that the collection of open circles also forms a basis for some topology on $\mathbb{R}^2$, which we have also used, that topology is denoted by $\mathcal{T}_d$. Finally, the collection of the diamond types of structures also forms a basis for some topology on $\mathbb{R}^2$, and that topology, we have denoted by $\mathcal{T}_{d_T}$. The question is, whether these are the same or different, or we can compare them something like one topology is coarser than another. Interestingly, it can be seen that all the topologies are the same, and the topology that we are getting here will be Euclidean topology on $\mathbb{R}^2$. Now, from here, what can we conclude? We can conclude that the geometric shapes of the individual basis element are not determining the open sets in the topology. The answer is simple behind this conclusion, because what we are looking at, or what we are observing that the different shapes something like open rectangles, open circles, and diamond type of structures individually, are bases for some topology on $\mathbb{R}^2$, but finally, the topology is going to be same.

Let us justify how the topology $\mathcal{T}_d$ induced by metric d and topology induced by taxicab metric, that is, $\mathcal{T}_{d_T}$, are the same. If we want to justify that both the topologies are equal, we have to just show that this $\mathcal{T}_d$ is coarser than this $\mathcal{T}_{d_T}$, and also, we have to justify that this $\mathcal{T}_{d_T}$ is coarser than $\mathcal{T}_d$. Just recall the concept of basis and how to compare the topologies. If we want to justify that $\mathcal{T}_d$ is coarser than $\mathcal{T}_{d_T}$, we have to show that if we are taking the basis element for it, that is, let us take the basis element $B_d(x, r)$ for topology $\mathcal{T}_d$ centered at x with radius $r, (r > 0)$. Then there exists $r' > 0$ such that $B_{d_T}(x, r') \subseteq B_d(x, r)$.

Similarly, whenever we want to show the $\mathcal{T}_{d_T}$ is coarser than $\mathcal{T}_d$, we have to show that if we are taking the basis element for it, that is, let us take the basis element $B_{d_T}(x, r)$ for topology $\mathcal{T}_{d_T}$ centered at x with radius $r, (r > 0)$. Then there exists $r' > 0$ such that $B_d(x, r') \subseteq B_{d_T}(x, r)$. Let us justify these two one by one. Now, the first thing is the choice of r'. Let us take this $r' = \frac{r}{\sqrt{2}}$, and also, in order to justify first inclusion, let us take $y \in B_{d_T}(x, r')$. Then

$d_T(x,y) < r'$, or $|x_1 - y_1| + |x_2 - y_2| < r'$, or we can also say that $|x_1 - y_1| < r'$ and $|x_1 - y_1| < r'$. Now, if we are computing $d(x,y)$, this will be nothing but $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$; this is less than $\sqrt{r'^2 + r'^2} = r'\sqrt{2}$, or this is equal r. So, what we have shown from here is that $d(x,y) < r$, or $y \in B_d(x,r)$. Thus, we have justified the first inclusion.

Let us move ahead and try to justify that $\mathcal{T}_{d_T}$ is coarser than $\mathcal{T}_d$, that is, our motive is to justify that $B_d(x,r') \subseteq B_{d_T}(x,r)$, where $r > 0$ is given to us and we have to choose r'. The simple way for this choice is, take r' as $r/2$. Now, take $y \in B_d(x,r')$. It means that $d(x,y) < r'$, or $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < r'$. Now, if we are computing $|x_1 - y_1|$, this is nothing but the positive square root of $(x_1 - y_1)^2$ and $(x_1 - y_1)^2 \leq \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < r'$. So, finally, we have with us that $|x_1 - y_1| < r'$. Similarly, we can also conclude that $|x_2 - y_2| < r'$. Now, we are computing $|x_1 - y_1| + |x_2 - y_2|$, it will be less than $2r'$, which is nothing but r. Thus $d_T(x,y) < r$. Therefore, we conclude that $y \in B_{d_T}(x,r)$, and thus the required inclusion is ok. So, we have shown two inclusions. Thus, this topology induced by metric d is the same as that induced by taxicab metric. So, we will denote these topologies by $\mathcal{T}_e$, and this is nothing but Euclidean topology on $\mathbb{R}^2$. In the same fashion, one can show that the topology generated by the basis, which is a collection of open rectangles, is also this Euclidean topology.

Moving ahead, what we have seen is that we have taken three bases. What was the conclusion? The conclusion was that all the bases are generating the same topology. A collection of open rectangles, a collection of open circles, and a collection of diamond types of structures are providing Euclidean topology on $\mathbb{R}^2$. The question is whether we have a metric that can induce a topology on $\mathbb{R}^2$ different from the Euclidean topology. The answer is here.

If we are taking $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, and let us define the metric d as:

$$d(x,y) = \begin{cases} 1, & \text{if } x_1 \neq y_1 \text{ or } |x_2 - y_2| \geq 1 \\ |x_2 - y_2|, & \text{if } x_1 = y_1 \text{ and } |x_2 - y_2| < 1. \end{cases}$$

It can be seen that this d is a metric on $\mathbb{R}^2$, and if this is a metric, obviously, this will induce a topology on $\mathbb{R}^2$. Just think about it. Interestingly, one can see that the topology induced by this metric is strictly finer than the Euclidean topology on $\mathbb{R}^2$. I am not proving this one. Just leaving it to think.

These are the references.

That’s all from this lecture. Thank you.