Course Name: Essentials of Topology
Professor Name: S.P. Tiwari
Department Name: Mathematics & Computing
Institute Name: Indian Institute of Technology(ISM), Dhanbad
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Welcome to Lecture 14 on Essentials of Topology.

In this lecture too, we will continue the study of the basis for a topology. Begin with what we have studied in the previous lecture: if we have a topological space $(X, \mathcal{T})$ and a sub-collection $\mathcal{B}$ of $\mathcal{T}$, then $\mathcal{B}$ is a basis for topology $\mathcal{T}$ if some conditions are satisfied. Specifically, we have seen that for all $G \in \mathcal{T}$ and for all $x \in G$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq G$. We have also discussed a different problem too, i.e., instead of having a topological space, let us begin with a nonempty set X and a sub-collection of $P(X)$, say $\mathcal{B}$. The question is, can we put a topology on X? As per our discussion in the previous lecture, beginning with an arbitrary sub-collection, we cannot do so. But if we are putting some restrictions, we can find a topology on X, by using this $\mathcal{B}$. The conditions were something like (i) X should be expressed as the union of the members of $\mathcal{B}$, and (ii) some restriction on $B_1 \cap B_2$ will be required. What exactly is our requirement? Let us see through the diagram. Actually, if we are beginning with any X, and if we are taking members of $\mathcal{B}$, their union should always be equal to X. The second condition we require is if we are taking any B_1 and B_2 in $\mathcal{B}$ and let us take an element x from $B_1 \cap B_2$. What exactly do we require? There should be something like B_3 in $\mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Precisely, this is the statement of the theorem which says that beginning with an arbitrary $\mathcal{B}$, that is, a collection of some subsets of X, how this $\mathcal{B}$ or when this $\mathcal{B}$ will be the basis for some topology on X. The theorem is stated here.

Let X be a set, and $\mathcal{B}$ be an arbitrary collection of subsets of X. Then $\mathcal{B}$ is a basis for a topology on X iff

(i) $X = \cup\{B : B \in \mathcal{B}\}$; and

(ii) For all $B_1, B_2 \in \mathcal{B}$ and for all $x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Let us see the proof. So, first, we assume that $\mathcal{B}$ is a basis for a topology on X. Then, we have to show that the two criteria will be satisfied. Now, if $\mathcal{B}$ is a basis for a topology on X. Then as we know that X is an open set, and also, we know that every open set can be expressed as the union of members of basis. Therefore, $X = \cup\{B : B \in \mathcal{B}\}$. This is the proof of the first one. Now, let us see the second one. If we are taking any $B_1, B_2 \in \mathcal{B}$. Note that these B_1 and B_2 will also be the members of topology because if $\mathcal{B}$ is a basis for some topology $\mathcal{T}$ on X. Then what is this $\mathcal{B}$? This $\mathcal{B}$ is a sub-collection of $\mathcal{T}$, and if $B_1, B_2 \in \mathcal{T}$, what about $B_1 \cap B_2$? That is also a member of $\mathcal{T}$, and if $B_1 \cap B_2$ is a member of $\mathcal{T}$, recall the definition of basis again. For all $x \in B_1 \cap B_2$, there exists a member in basis, that is, $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. So, this is the proof of one part.

In order to prove the converse, let us assume that the given two conditions are satisfied. Then, our motive is to prove that $\mathcal{B}$ is the basis for a topology on X. However, we don't have any information about topology. What exactly we are going to do is use the concept of basis to try to put a topology on X. Actually, we will take a collection inspired by the definition of basis, and we will show that the collection is a topology on X. How to choose that collection? Let us take as:

$$\mathcal{T} = \{G \subseteq X : \text{for all } x \in G, \text{ there exists } B \in \mathcal{B} \text{ such that } x \in B \subseteq G\}.$$

The interesting feature of this collection is that if we can show that $\mathcal{T}$ is a topology on X. Then, obviously, $\mathcal{B}$ will be a basis for this topology. Now, let us show that this $\mathcal{T}$ is a topology on X. Note that, the empty set will always be a member of $\mathcal{T}$. Also, we can show that $X \in \mathcal{T}$ because if we are taking any $x \in X$. Then as $X = \cup\{B : B \in \mathcal{B}\}$. Meaning is, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq X$. Thus, X is a member of $\mathcal{T}$.

Moving ahead, let us show that if $G_1, G_2, ..., G_n$, that is we are taking a finite number of members in $\mathcal{T}$, and show that their intersection, that is $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$. But what we are going to do exactly is to show that if $G_1, G_2 \in \mathcal{T}$, $G_1 \cap G_2 \in \mathcal{T}$, and by induction, we can also prove $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$. In order to prove it, if we are taking $G_1, G_2 \in \mathcal{T}$, let us take any $x \in G_1 \cap G_2$. What will happen? $x \in G_1$, and $x \in G_2$. Note that if $x \in G_1$, there exists $B_1 \in \mathcal{B}$, satisfying the criteria that $x \in B_1 \subseteq G_1$. Similarly, if $x \in G_2$, there exists some $B_2 \in \mathcal{B}$, such that $x \in B_2 \subseteq G_2$. Combining these two, we can conclude that $x \in B_1 \cap B_2 \subseteq G_1 \cap G_2$. Now, coming to the second assumption, which says that if $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. So, what we can say is that there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$, that is a subset of $G_1 \cap G_2$, or $x \in B_3 \subseteq G_1 \cap G_2$, and therefore $G_1 \cap G_2 \in \mathcal{T}$. Thus by induction, we can also conclude it.

Moving to the next, let us take an arbitrary family from $\mathcal{T}$, i.e., $\{G_i : i \in I\}$, where I is an index set, and $G_i \in \mathcal{T}$. What is our motive? We have to justify that their union, that is, $\cup\{G_i : i \in I\} \in \mathcal{T}$. How do you show it? So, begin with an element $x \in \cup\{G_i : i \in I\}$. Then $x \in G_i$, for some $i \in I$. It is to be noted that $G_i \in \mathcal{T}$. Now, if $G_i \in \mathcal{T}$, what can we conclude? We can conclude that there is some $B \in \mathcal{B}$ such that $x \in B \subseteq G_i$. But at the same time, it is to be noted that $G_i \subseteq \cup\{G_i : i \in I\}$, that is, $x \in B \subseteq \cup\{Gi : i \in I\}$. Thus, we can conclude that $\cup\{Gi : i \in I\} \in \mathcal{T}$, and hence $\mathcal{T}$ is a topology on X. If $\mathcal{T}$ is a topology on X, then the collection $\mathcal{B}$ is a basis for this topology, that is obvious from the definition itself. Moving ahead, let us take some of the examples.

The first example is, let us take a set $X = \{a, b, c\}$. Also, let $\mathcal{B} = \{\{a, b\}, \{b\}, \{b, c\}\}$. It is to be noted that X can be expressed as the union of members of $\mathcal{B}$; that is, if we are taking even $\{a, b\} \cup \{b, c\}$, there is no need to take the singleton set $\{b\}$. Further, if we are taking any B_1 and B_2, say $B_1 = \{a, b\}$ and $B_2 = \{b\}$, or if we are taking B_1 as $\{a, b\}$ and B_2 as $\{b, c\}$, even if we are taking B_1 as $\{b\}$ and B_2 as $\{b, c\}$, what about their intersections? Their intersection is nothing, but this is a singleton set $\{b\}$, in all the cases. So, take $\{b\}$ as B_3, which is in $\mathcal{B}$. Therefore, $\mathcal{B} = \{\{a, b\}, \{b\}, \{b, c\}\}$ will generate a topology on X, and if we want to see what topology that is? This is $\mathcal{T} = \{\emptyset, X, \{a, b\}, \{b\}, \{b, c\}\}$.

In the next example, why not let us take X as a nonempty set and now let us take a collection $\mathcal{B} = \{\{x\} : x \in X\}$. We can justify that this $\mathcal{B}$ is also a basis for some topology on X, and that topology is well known. The first one is, it is to be noted that X can always be expressed as the union of these singleton sets $\{x\}$, where $x \in X$, and the second one, if we are taking any two elements, that is singleton set $\{x\}$ and singleton set $\{y\}$, these two are in $\mathcal{B}$. If x is not equal to y, what will happen? Actually, $\{x\} \cap \{y\} = \emptyset$, and in this case, the second condition will be satisfied trivially. Thus, this will generate a topology on X, and the topology is well known to us, that topology is nothing but discrete topology.

Moving ahead, let us take another example: we have already seen the basis for Euclidean topology on the set of reals. We have also seen the basis for this lower limit topology, even the basis for an upper limit topology on the set of real numbers. Let us take another collection, that is $\mathcal{B}$, which is given as

$$\mathcal{B} = \{(a,b) : a, b \in \mathbb{R}\} \cup \{(a,b) - K : a, b \in \mathbb{R}\}, K = \{\tfrac{1}{n} : n \in \mathbb{N}\}.$$

We can show that this $\mathcal{B}$ will also generate a topology on the set of real numbers.

Begin with, let us see whether $\mathbb{R}$ can be expressed as the union of members of this $\mathcal{B}$. The answer is yes, because what $\mathcal{B}$ we have taken, that is a collection of open intervals plus some additional sets. Also, we know that the collection of open intervals forms a basis for Euclidean topology on the set of real numbers. So, obviously, the union of members of $\mathcal{B}$ will always be equal to $\mathbb{R}$. Coming to the second one, in this case, there will be different cases.

Let us discuss one by one. So the first case is, let us take this case as (a). So, let $B_1 = (a,b)$, and $B_2 = (c,d)$. Now, if we are finding out $B_1 \cap B_2$, and $B_1 \cap B_2$ is the empty set, the things will be satisfied trivially. But if $B_1 \cap B_2$ is nonempty, let us think about it. Now, let us take $x \in B_1 \cap B_2$. Then what will happen? Actually, $x \in B_1 = (a,b)$, and $x \in B_2 = (c,d)$. From these two, it is clear that $a < x < b$ and $c < x < d$. Now, let us take $e = max\{a,c\}$ and $f = min\{b,d\}$. Then $x \in (e,f) \subseteq (a,b) \cap (c,d)$. Also, it is to be noted that $B_3 = (e,f) \in \mathcal{B}$.

Moving ahead, let us take a look at the second case. What are we taking? We are taking $B_1 = (a,b)$, and let us take $B_2 = (c,d) - K$. Now, let us discuss about $B_1 \cap B_2$. What is it? This is nothing but $(a,b) \cap ((c,d) - K) = (a,b) \cap ((c,d) \cap K^c) = ((a,b) \cap (c,d)) \cap K^c$. We have already seen that if (a,b) and (c,d) are not disjoint, then $((a,b) \cap (c,d)) \cap K^c$ can be written as $(e,f) \cap K^c = (e,f) - K$, and it is to be noted that this is a member of $\mathcal{B}$. So, what we can do that, we can take $B_3 = (e,f) - K$, and our problem get solved.

Coming to the third case. Let us take $B_1 = (a,b) - K$ and $B_2 = (c,d) - K$.

Then $B_1 \cap B_2 = (a, b) \cap (c, d) \cap K^c$. It is to be noted that if (a, b) and (c, d) are not disjoint, then $B_1 \cap B_2 = (e, f) \cap K^c$, or $B_1 \cap B_2$ can be written as $(e, f) - K$, and that is in $\mathcal{B}$. So, we can assume $(e, f) - K$ as B_3, and therefore the second condition will be satisfied. So, what have we shown? If we are beginning with the given $\mathcal{B}$, the set of real numbers can be expressed as the union of members of this $\mathcal{B}$, and if we are taking any $B_1, B_2 \in \mathcal{B}$, and if we are taking any $x \in B_1 \cap B_2$, we have shown that in all cases there exists a $B_3 \in \mathcal{B}$ so that $x \in B_3 \subseteq B_1 \cap B_2$. Thus, $\mathcal{B}$ will be a basis for some topology on $\mathbb{R}$, and that topology is known as K-topology. We will denote this topology by $\mathcal{T}_K$.

A question will come if this is a new topology on the set of real numbers, can we compare this topology with the well-known topologies on the set of real numbers, such as Euclidean topology, lower limit topology, and upper limit topology? Just think about it. Moving ahead, we have already studied the concept of comparison of topologies, and that comparison of topologies was given in terms of open sets, that is, if we are taking two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, if every $\mathcal{T}_1$-open set is a $\mathcal{T}_2$-open set, we say that the topology $\mathcal{T}_1$ is coarser than the topology $\mathcal{T}_2$. Can we compare the topologies by using basis? The answer is yes, and that is given here in the form of a result: Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for topologies $\mathcal{T}_{\mathcal{B}_1}$ and $\mathcal{T}_{\mathcal{B}_2}$ on a nonempty set X. Then $\mathcal{T}_{\mathcal{B}_1} \subseteq \mathcal{T}_{\mathcal{B}_2}$ iff $\forall x \in X$ and $\forall B_1 \in \mathcal{B}_1$ containing x, there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.

Let us prove this result. So, first, what are we doing? We are assuming that the topology $\mathcal{T}_{\mathcal{B}_1}$ is coarser than the topology $\mathcal{T}_{\mathcal{B}_2}$. Also, let us take $B_1 \in \mathcal{B}_1$, and $x \in B_1$. Our motive is to search for some $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$. Now, it is to be noted that $B_1 \in \mathcal{B}_1$, and what is this $\mathcal{B}_1$? Note that the basis is always a sub-collection of a topology generated by it. So, $\mathcal{B}_1$ is a sub-collection of $\mathcal{T}_{\mathcal{B}_1}$. But note that $\mathcal{T}_{\mathcal{B}_1}$ is coarser than $\mathcal{T}_{\mathcal{B}_2}$. What does it mean? It means that B_1 is a member of topology $\mathcal{T}_{\mathcal{B}_2}$ and this topology is generated by a basis, and what is the basis? That basis is nothing but $\mathcal{B}_2$. Now, recall the definition of basis. It says that if $x \in B_1$ and $B_1 \in \mathcal{T}_{\mathcal{B}_2}$, there exists some element in the basis for this topology $\mathcal{T}_{\mathcal{B}_2}$, and that basis is $\mathcal{B}_2$. Thus there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$. So, it proves our requirement.

Moving to the converse part of this result. Now, let us assume that $\forall x \in X$ and $\forall B_1 \in \mathcal{B}_1$ containing x, there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$. Then our motive is to justify that the topology $\mathcal{T}_{\mathcal{B}_1}$ generated by basis $\mathcal{B}_1$ is coarser than the topology $\mathcal{T}_{\mathcal{B}_2}$, which is generated by basis $\mathcal{B}_2$. In order to justify it, let us take $G \in \mathcal{T}_{\mathcal{B}_1}$. Now, we are taking any $x \in G$. What do we have by definition of basis? There exists some $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq G$. But at the same time, what is our assumption? If B_1 contains x, there exists $B_2 \in \mathcal{B}_2$ satisfying $x \in B_2 \subseteq B_1$, or $x \in B_2 \subseteq G$. Note that this is true for all $x \in G$. So what we can deduce from here that G can be expressed as union of some members of $\mathcal{B}_2$. It is to be noted that all the members of $\mathcal{B}_2$ will also be members of topology $\mathcal{T}_{\mathcal{B}_2}$. Therefore, their union, that is, G is also a member of topology $\mathcal{T}_{\mathcal{B}_2}$, or that the topology generated by basis $\mathcal{B}_1$ is coarser than the topology generated by basis $\mathcal{B}_2$.

These are the references.

That's all from this lecture. Thank you.