Course Name: Essentials of Topology
Professor Name: S.P. Tiwari
Department Name: Mathematics & Computing
Institute Name: Indian Institute of Technology(ISM), Dhanbad
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Welcome to Lecture 13 on Essentials of Topology.

In this lecture, we will study the concept of the basis for a topology. Roughly speaking, it is a smaller structure than topology, which can generate the given topology. Let us see the motivation behind the concept. Begin with the concept of Euclidean topology on the set of real numbers, which we have already seen.

$$\mathcal{T}_e = \{G \subseteq \mathbb{R} : \forall x \in G, \exists a, b \in \mathbb{R} \text{ with } a < b \text{ such that } x \in (a, b) \subseteq G\}.$$

The question is, can we characterize the open sets G in some other way? The answer is, G is an open set if and only if G is the union of open intervals. This is another characterization of open sets for Euclidean topology. Let us see this result first.

Begin with, let us consider G as an open set. Then, for all $x \in G$, from the definition of topology, there exist $a_x, b_x \in \mathbb{R}$ with $a_x < b_x$ such that $x \in (a_x, b_x) \subseteq G$. Now, we can see that $G = \cup_{x\in G}(a_x, b_x)$. If it is so, then we proved the requirement. The question is, is it possible? The answer is yes. Why? Because if we are taking $y \in G$, then as G is open, there exist $a_y, b_y \in \mathbb{R}$ with $a_y < b_y$ such that $y \in (a_y, b_y) \subseteq G$. Therefore, $y \in \cup_{x\in G}(a_x, b_x)$, that is, $G \subseteq \cup_{x\in G}(a_x, b_x)$. Coming to the rest part, if we are taking $y \in \cup_{x\in G}(a_x, b_x)$. Then there exists an open interval (a', b') such that $y \in (a', b') \subseteq G$, i.e., $y \in G$, or that $\cup_{x\in G}(a_x, b_x) \subseteq G$. Thus, we have shown that $G = \cup_{x\in G}(a_x, b_x)$.

Now, let us see the converse part of this result. If G is the union of open intervals, that is, $G = \cup_{i\in I}(a_i, b_i)$, and I is an index set. Then, if we are taking any $x \in G$, what will happen? $x \in \cup_{i\in I}(a_i, b_i)$, or x belongs to at least one interval, that is (a_k, b_k), which is a subset of $\cup_{i\in I}(a_i, b_i)$, and that is precisely G. So, finally we can see that $x \in (a_k, b_k) \subseteq G$. Note that these a_k, b_k are from the set of reals, and $a_k < b_k$. Therefore, G is an open set.

Thus, what we can conclude from here is that the open sets can be characterized in terms of open intervals, or instead of writing distinct members of the topology, it is sufficient to say that the collection of open intervals generates this topology. So, this is actually the basic motivating factor regarding the concept of basis for a topology. Formally, the basis for topology is given like this one: a collection $\mathcal{B}$ of open subsets of X is called a basis for topology $\mathcal{T}$ if every open set is a union of members of $\mathcal{B}$.

So, one can see, or one can relate that what we have seen in the case of Euclidean topology and the characterization of an open set is motivating the definition of the basis for a topology. Let us see some of the examples. Beginning with the first one, if we are taking $X = \{a, b, c, d\}$. Also, let us take a topology $\mathcal{T} = \{\emptyset, X, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}\}$ on it. We can see that $\mathcal{B} = \{\{a\}, \{b, c\}, \{d\}\}$ is a basis for this topology. The question is, how? The answer is that empty set can be written as an empty union of members of $\mathcal{B}$. Moving ahead, X can be written as a union of $\{a\}, \{b, c\}$, and $\{d\}$. Note that $\{a\}, \{b, c\}$, and $\{d\}$ are already in $\mathcal{B}$. If we are looking for $\{a, b, c\}$, note that $\{a, b, c\}$ is the union of two sets, that is, the singleton set $\{a\}$ and $\{b, c\}$. Similarly, if we are looking for $\{a, d\}$, this is a union of $\{a\}$ as well as $\{d\}$, and finally, if we are coming for $\{b, c, d\}$, that is the union of $\{b, c\}$ as well as $\{d\}$. So, all the elements of $\mathcal{T}$ can be expressed as the union of members from $\mathcal{B}$, that is why $\mathcal{B}$ is a basis for $(X, \mathcal{T})$. It is okay that all the members of topology can be expressed as the union of members from $\mathcal{B}$, but at the same time it can be seen that this $\mathcal{B}$ is nothing, but this is a collection of open sets, because singleton set $\{a\}$ is in topology, two elements set $\{b, c\}$ is in the topology, and singleton set $\{d\}$ is also in the topology.

Moving ahead, let us come to the well-known example, that is the discrete topological space $(X, \mathcal{T})$. In discrete space, we already know that the discrete topology $\mathcal{T}$ is nothing but the power set of X. If we are taking empty set that will be the empty union of members of $\mathcal{B} = \{\{x\} : x \in X\}$, but if we are taking any non-empty G, which is in the discrete topology, then this G can be written as $\cup_{x \in G}\{x\}$. It means that, all the open sets can be expressed as the union of members of $\mathcal{B}$, and that's why $\mathcal{B} = \{\{x\} : x \in X\}$ is a basis for the discrete topology on X.

Now, we are going to characterize the basis in some different way, and this is also a simple one. What the characterization is, that if $(X, \mathcal{T})$ be a topological space. Then a collection $\mathcal{B}$ of open subsets of X is a basis for topology $\mathcal{T}$ iff for all open set G and for all $x \in G$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq G$. This is simple to deduce.

Begin with, let $\mathcal{B}$ be a basis for topology $\mathcal{T}$ and G be an open set. Then, from the definition of topology, G can be expressed as the union of some members of basis $\mathcal{B}$, that is, $G = \cup\{B_i : i \in I\}$, I is an indexed set, where $B_i \in \mathcal{B}$. If we are now taking any $x \in G$, then $x \in \cup\{B_i : i \in I\}$. Note that every B_i is in $\mathcal{B}$, meaning is, x belongs to at least one B_k, and $k \in I$. From here, we can also write that $x \in B_k \subseteq \cup\{B_i : i \in I\}$, and that is nothing but G. Therefore, $x \in B_k \subseteq G$. So, we have shown that for every open set G and for all elements $x \in G$, there exists $B \in \mathcal{B}$, and that B is precisely B_k such that $x \in B_k \subseteq G$. Coming to the converse of this result. Now, let us assume that for all open set G and for all $x \in G$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq G$. If we are assuming it, our motive is to justify that this $\mathcal{B}$ will be the basis for topology. So, let us take an open set G. Then, for all $x \in G$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq G$. Now, this G can be written as the union of $\{B_x : x \in G\}$. It means that G can be expressed as the union of members of $\mathcal{B}$; that is why $\mathcal{B}$ is a basis for topology. So, the conclusion is that the basis for a topology can also be characterized by using this concept. Let us use this concept and see some examples of the basis.

Begin with the well known Euclidean topology $\mathcal{T}_e = \{G \subseteq \mathbb{R} : \forall x \in G, \exists a, b \in \mathbb{R}$ with $a < b$ such that $x \in (a, b) \subseteq G\}$. Now, if we are taking $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$. Then, if we compare with the characterization of basis and look carefully at the definition of topology, we can conclude that $\mathcal{B}$ is a basis for Euclidean topology. Moving ahead, let $\mathcal{T}_l = \{G \subseteq \mathbb{R} : \forall x \in G, \exists a, b \in \mathbb{R}$ with $a < b$ such that $x \in [a, b) \subseteq G\}$. Then $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}\}$ is a basis for topology $\mathcal{T}_l$, that is, the lower limit topology.

In the same fashion, coming to upper limit topology $\mathcal{T}_u = \{G \subseteq \mathbb{R} : \forall x \in G, \exists a, b \in \mathbb{R}$ with $a < b$ such that $x \in (a, b] \subseteq G\}$ and compare with the characterization of a basis, then what is happening here? The collection, that is, $\mathcal{B} = \{(a, b] : a, b \in \mathbb{R}\}$ is a basis for upper limit topology. Moving ahead, why not let us discuss about metric topology? So, for a metric space (X, d), actually metric topology $\mathcal{T}_d = \{G \subseteq X : \forall x \in G, \exists r > 0$ such that

$B(x, r) \subseteq G\}$ is well known to us, and if we are comparing the definition of basis and the concept of open set in metric topology, we can conclude that $\mathcal{B} = \{B(x, r) : x \in X, r > 0\}$, is a basis for this metric topology.

Now, let us see again the example which we have seen earlier on the set $X = \{a, b, c, d\}$ with topology $\mathcal{T} = \{\emptyset, X, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}\}$. We have seen that $\mathcal{B} = \{\{a\}, \{b, c\}, \{d\}\}$ is a basis for this topology. The question is, is it only the basis for this topology? The answer is that we can construct some more, too. So, what we are taking, let us take another $\mathcal{B} = \{\{a\}, \{b, c\}, \{d\}, \{a, d\}\}$. Then this is also a basis for topology. The answer is simple. Why is this happening? If all $\mathcal{T}$-open sets can be expressed as the union of members of $\mathcal{B} = \{\{a\}, \{b, c\}, \{d\}\}$, then obviously, the $\mathcal{T}$-open sets can also be expressed as the union of members of $\mathcal{B} = \{\{a\}, \{b, c\}, \{d\}, \{a, d\}\}$. We can add some more elements here, and that will still be a basis. Let us see this is $\mathcal{B} = \{\{a\}, \{b, c\}, \{d\}, \{a, d\}, \{a, b, c\}\}$. We can again check that this is also a basis for the topology $\mathcal{T}$, because it is consisting the elements which are sufficient to justify that it is a basis plus, we are adding some additional sets. But note that these additional sets are open, too.

The final conclusion from this example is that, corresponding to a topology, we can construct a number of bases, which we have already seen. Coming to the example of Euclidean topology. We have seen that $\mathcal{B}_1 = \{(a, b) : a, b \in \mathbb{R}\}$ is a basis for Euclidean topology. Instead of taking a and b as real numbers, if we are taking a and b as rational numbers, the question is, is $\mathcal{B}_2 = \{(a, b) : a, b \in \mathbb{Q}\}$ a basis for Euclidean topology? The answer is yes. Why is this possible? Because if we are looking at $\mathcal{B}_1 = \{(a, b) : a, b \in \mathbb{R}\}$, what is happening that for every G in the Euclidean topology and for all $x \in G$, we are getting two real numbers, a and b, with $a < b$, such that $x \in (a, b) \subseteq G$. In between a and x, by using the denseness of rationals, we can find a rational number c, and a rational number d in between x and b such that $x \in (c, d) \subseteq (a, b) \subseteq G$. Note that c and d are now rational numbers. So, there exist rational numbers c and d, $c < d$ such that $x \in (c, d) \subseteq G$. So, from here, we can say that $\mathcal{B}_2$ is also a basis for Euclidean topology.

Coming to the next one, from the previous discussion, we have seen that there are different bases for a topology. The question is how to find out. The idea is here: Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be a basis for topology $\mathcal{T}$. Then all $\mathcal{B}'$ which satisfying the relationship, that is, $\mathcal{B}'$ is a sub-collection of $\mathcal{T}$ and contains $\mathcal{B}$ is a basis for topology $\mathcal{T}$. The question is, how is it possible? The answer is simple because if we are taking any G in the topology $\mathcal{T}$, if $\mathcal{B}$ is the basis for it, then $G = \cup\{B_i : i \in I\}$. It is to be noted that B_i is a member of $\mathcal{B}$, and if B_i is a member of $\mathcal{B}$, because $\mathcal{B}$ is a sub-collection of $\mathcal{B}'$, so B_i is also a member of $\mathcal{B}'$, that is, $G = \cup\{B_i : i \in I\}, B_i \in \mathcal{B}'$. Thus, $\mathcal{B}'$ is a basis for topology $\mathcal{T}$. It is to be noted from here that every topology is a basis for itself, that is, $\mathcal{T}$ is a basis for itself. The question is whether the basis is also a topology. The answer is that the basis may not be a topology. We have seen a number of examples.

Moving ahead, what we have seen is that a topology may have different bases. The question comes whether a sub-collection $\mathcal{B}$ can be the basis for two different topologies, that is, this $\mathcal{B}$ is a basis for both the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. The answer is no, and the statement is here. If $\mathcal{B}$ is a basis for the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X. Then both the topologies will always be the same; that is, the topology generated by basis will always be unique. In order to justify it, if $\mathcal{B}$ is a basis for the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, then what will happen? This $\mathcal{B}$ is a sub-collection of $\mathcal{T}_1$, and this $\mathcal{B}$ is also a sub-collection of $\mathcal{T}_2$. Our motive is to justify that the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are the same. In order to justify it, we have to justify two things: the topology $\mathcal{T}_1$ is coarser than the topology $\mathcal{T}_2$, and topology $\mathcal{T}_2$ is coarser than the topology $\mathcal{T}_1$. Let us explain the first one. If we are taking any $G \in \mathcal{T}_1$, by the definition of basis, $G = \cup\{B_i : i \in I\}$. But note that $B_i \in \mathcal{B}$. Now, as $\mathcal{B}$ is a sub-collection of topology $\mathcal{T}_2$, what will happen? It means that this B_i is also a member of topology $\mathcal{T}_2$. Now, by the definition of topology, $\cup\{B_i : i \in I\}$, is also in $\mathcal{T}_2$, that is, G is a member of $\mathcal{T}_2$. So, from here, we can conclude that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$. Similarly, one can justify that $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$. Hence $\mathcal{T}_1$ and $\mathcal{T}_2$ are the same.

Finally, what conclusion do we have with us regarding basis and topology? The answer is here. If this is a collection of bases and this is a collection of topologies. Now, if we are having some $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ here and we are having a topology $\mathcal{T}$, $\mathcal{B}_1$ may be the basis for topology $\mathcal{T}$, $\mathcal{B}_2$ may be the basis for this topology $\mathcal{T}$ and $\mathcal{B}_3$ may also be the basis for the same topology. But if we are having some basis $\mathcal{B}$, the question is, is it possible that this is the basis for two topologies? This is not possible.

Moving ahead, let us discuss it in a different way. What have we seen till now is that if a topological space is known to us, how do we construct a basis for it? We know that the open sets can be expressed as the union of members of basis. In other direction, if this is given to us that we have a non-empty set X, we have some collection $\mathcal{B}$, that is, $\mathcal{B}$ is a subset of the power set of X. The question is, whether $\mathcal{B}$ is a basis for some topology on X? Let us see through an example.

For example, if we are having a non-empty set X, let us take this is a collection $\{a, b, c, d\}$. Also, let $\mathcal{B}$, which is a subset of the power set of X, and this is $\{\{a, b\}, \{b, c\}\}$. The question is whether $\mathcal{B}$ can be a basis for some topology on X. Let us see. Recall the relationship between basis and topology that the open sets should be expressed as a union of members of $\mathcal{B}$. So, if we want to construct some $\mathcal{T}$, it will contain the empty set, $\{a, b\}$, $\{b, c\}$ and $\{a, b, c\}$. The question is, is it a topology? The answer is no. There are two problems. What is the first problem? The first problem is that X is not a member of this $\mathcal{T}$, and what is the second problem? The second problem is that sets $\{a, b\}$ and $\{b, c\}$ both are in $\mathcal{T}$; what about their intersection? That is $\{a, b\} \cap \{b, c\}$, that will be a singleton set $\{b\}$, that is not in $\mathcal{T}$. So, if we are beginning with an arbitrary set and we are saying that this set will be the basis for some topology, the answer is no.

Actually, $\mathcal{B}$ should have some properties. From above, it is clear that X should be in $\mathcal{T}$. How will X come to $\mathcal{T}$? If X can be expressed as a union of members of $\mathcal{B}$. So, our choice of $\mathcal{B}$ should be in such a fashion that X can be expressed as a union of members of $\mathcal{B}$, which was not here. The second thing is if we are taking some elements in this $\mathcal{B}$, there will be some characterization regarding their intersection. So, $\mathcal{B}$ will be the basis for some topology if we put some restrictions on $\mathcal{B}$. The first restriction is regarding the union of members should be equal to X, and the second restriction is regarding their intersection. If we are putting these two restrictions, we can begin with an arbitrary $\mathcal{B}$ with these restrictions and we can show that $\mathcal{B}$ is a basis for some topology.

These are the references.

That's all from this lecture. Thank you.