

Course Name: Essentials of Topology
Professor Name: S.P. Tiwari
Department Name: Mathematics & Computing
Institute Name: Indian Institute of Technology(ISM), Dhanbad
Week: 02
Lecture: 05

Welcome to Lecture 12 on Essentials of Topology.

In the previous lectures, we have studied the concept of open sets. In this lecture, we will study the complementary notion of open sets, known as closed sets. The definition is simple, but we will see the importance of this concept in some notions available in the topological spaces. Formally, a closed set is defined as: Let (X, \mathcal{T}) be a topological space. Then a subset F of X is called a closed set if F^c is open.

From the definition, it seems that if a set is closed, then it cannot be open, or a set that is open cannot be closed. But we will see the interesting features of closed and open sets. We can get a number of sets that are neither open nor closed or can be closed and open simultaneously.

Let us take some of the examples. This example is from Euclidean topology. Let us check whether the intervals, that are closed interval $[a, b]$ and these semi-open intervals $[a, b)$, $(a, b]$ are closed in the Euclidean topology, or not. Begin with the closed interval $[a, b]$. If we are computing $[a, b]^c$, it is simple that $[a, b]^c = (-\infty, a) \cup (b, \infty)$. We know that the intervals $(-\infty, a)$ and (b, ∞) are open in Euclidean topology. Therefore, their union is also open, that is, $[a, b]^c$ is in Euclidean topology or open in this topology. Thus, the closed interval $[a, b]$ is a closed set.

Let us talk about the semi-open interval $[a, b)$. Note that, $[a, b)^c = (-\infty, a) \cup [b, \infty)$. This is not a member of the Euclidean topology, because of the closedness of interval at b . Therefore, this semi-open interval is not a closed set. Similarly, if we take the semi-open interval $(a, b]$. Then $(a, b]^c = (-\infty, a] \cup (b, \infty)$. Because of a , the interval $(-\infty, a]$ will not be a member of the Euclidean topology. Therefore, we can conclude that $(a, b]$ is not a closed set.

So, we took three intervals and checked their behavior to determine whether

these intervals were closed in Euclidian topology or not. But, if we are making a simple change in the topology, instead of taking the Euclidean topology, let us take the lower limit topology. Now, check whether these three intervals are closed in the lower limit topology. For $[a, b]$, the answer is yes because it's complement is also open in the lower limit topology. Therefore, $[a, b]$ is a closed set in the lower limit topology. If we are coming to $[a, b)$, in the lower limit topology. Then as $[a, b)^c = (-\infty, a) \cup [b, \infty)$, which is the union of two open sets of the lower limit topology. Therefore, $[a, b)$ is a closed set. Note that $[a, b)$ is closed in lower limit topology, but not in the Euclidean Topology. If we are coming to $(a, b]$, the problem will be at a , because $(-\infty, a]$ will not be a member of the lower limit topology. Therefore, we can conclude that $(a, b]$ is not a closed set. Similarly, for these intervals, we can check whether these are closed in the upper limit topology or not.

Moving ahead, we come back to Euclidean topology. All singleton sets are closed in the Euclidean topology. Because, if we take any real number x and find $\{x\}^c$, that will be nothing but $(-\infty, x) \cup (x, \infty)$. Also, these two intervals are members of Euclidean topology, and so their union. That is why this singleton set $\{x\}$ is a closed set.

Coming to the concept of a finite set. Let us take a finite set $A = \{x_1, x_2, \dots, x_n\}$, where x_1, x_2, \dots, x_n are real numbers. Can we talk about the ordering between them? Yes. So, let $x_1 < x_2 < \dots < x_n$. Now, we are looking for A^c , which we have seen in the previous lecture in the case of co-finite topology. This will be $(-\infty, x_1) \cup (x_1, x_2) \cup \dots \cup (x_n, \infty)$. This union will always be a member of the Euclidean topology. Therefore, we can say that A is a closed set.

Coming to the more general case, that is whether \mathbb{Z} is a closed set in Euclidean topology. Note that, $\mathbb{Z}^c = \cup_{n \in \mathbb{Z}} (n, n + 1)$. This is inspired from the fact that whenever we are talking about \mathbb{Z}^c , we have to exclude all the integers, and therefore, we have to include only open intervals. So, if we are taking their union, that is the union of open intervals $(n, n + 1), n \in \mathbb{Z}$. Thus \mathbb{Z}^c is a member of the Euclidean topology. Therefore, \mathbb{Z} is a closed set.

Moving ahead, let us talk about the set of rationals \mathbb{Q} and the set of irrationals \mathbb{Q}^c . Note that none of these can be a closed set. The question is, why? Because if we are talking about \mathbb{Q}^c , that is not open in the Euclidean topology,

which we have already seen. Similarly, if we are taking \mathbb{Q}^c and we are taking its complement, that will be the set of rational numbers, and that is not a member of the Euclidean topology. Therefore, none of these are closed sets.

Coming to the next examples, the well-known indiscrete and discrete topological spaces. Let us see the concept of closed sets in these two. If we are having a non-empty set X with the indiscrete topology \mathcal{T} on it. It is well-known that \mathcal{T} is a collection of empty set as well as the set X . If we are looking for the complements, that is X and empty set. Note that X and empty set, both are open sets as well as closed sets in the indiscrete topology. It is to be pointed out here that if $A \subseteq X$, is both open and closed, we call such a set, a clopen set.

Moving to the next one, that is the concept of discrete topological spaces. As we already know, the discrete topology on a set X is nothing but the power set of X , which means all the subsets of X are open. If we are finding out the complement of subsets, that will always be subsets of X . Even, we will get all subsets of X , meaning is, all subsets of X are closed.

So, discrete topological space has this interesting feature: all the subsets are open and closed. Moving to the next one, what have we seen from these two examples? Note that the sets that are open here are closed, too. In the case of discrete topology, the sets that are open are also closed. A question arises whether these are the only two topologies with this feature that the open subsets are closed. The answer is no. We can construct a number of such topologies.

For example, if we are taking a set $X = \{a, b, c, d\}$. Let us take a collection $\mathcal{T} = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. It is clear that \mathcal{T} is a topology on X . If we are finding out the complement of these sets, then what we are getting, that is X , \emptyset , $\{c, d\}$ and $\{a, b\}$. The open sets, i.e., empty set and X are closed as their complement is open. Also, $\{a, b\}$ and $\{c, d\}$ are both open and closed. So, beyond indiscrete and discrete topologies, we can construct other topologies having the feature that the sets which are open are also closed.

Coming to one more feature of discrete topology. Discrete topology, as we know, can be characterized by using the concept of open sets; that is, a topology is discrete if all of its subsets are open. The question is, can we also

describe the discrete topology as a topology having all the subsets closed? Meaning is, if every subset is closed whether the topology is discrete? The answer is yes. We can do this in the case of discrete topology.

Let us take one more example. Take a set X as $\{a, b, c, d\}$. Put a topology $\mathcal{T} = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$ on it. Now, the closed sets are X , \emptyset , $\{b, c, d\}$, $\{a, d\}$, $\{d\}$, and $\{a\}$. Let us make some analysis about these sets. What about the singleton set $\{a\}$? This set is open and closed because it is in topology, and its complement is $\{b, c, d\}$, which is a member of \mathcal{T} , that's why this is both open and closed. Let us take the set $\{a, c, d\}$. Note that this is not in the topology, and if we are taking its complement, that will be the singleton set $\{b\}$, meaning is, this is neither open nor closed. Moving ahead, let us discuss about the set $\{b, c\}$. This is in the topology, so this is open but not closed. If we are taking the singleton set $\{d\}$, $\{d\}^c$ is $\{a, b, c\}$, which is open that's why $\{d\}$ is closed, but this singleton set $\{d\}$ is not a member of topology. So, this is not open but closed. Thus, this is interesting in the case of a topology that we can get a set that is both open as well as closed. We can also get the subsets which are neither open nor closed. We can also get the sets that are open but not closed, and we can also get the sets that are not open but closed. Meaning is, there are different possibilities in the case of topology regarding openness and closedness of a subset. So, from here, we can think about how a subset differs from a door.

A door must be open or closed, but note that a subset can be either open or closed. It may be both open and closed, and it may happen that this is neither open nor closed. It is an interesting one. It is hard to resist math humor. We will see these concepts later on again.

Coming to the next example, the concept of closed sets in a co-finite topological space. Here is a simple characterization, that is a subset F of X is closed if and only if it is finite. We can easily deduce it because if we are talking about the closedness of F , that is, F is closed means that F^c is open, and F^c is open, is equivalent to say that F^c is a member of this co-finite topology. If this is a member of the co-finite topology, its complement is finite, and the complement is finite, meaning F itself is finite.

So, the idea is, in a co-finite topological space, closed sets are nothing but

finite sets (obviously, X is closed too). Similar to the concept of co-finite topology, there is one well-known topology, that is co-countable topology. Here, a closed subset of X can also be characterized as a countable set. In the case of co-finite topology, we have seen that a subset F of X is closed if and only if it is finite, but as we know how co-countable topology differs from co-finite topology, that's why a subset F of X is closed if and only if it is countable (obviously, X is closed too).

Moving ahead, these are some of the properties of closed sets; the empty set and X are always closed sets. The finite union of closed sets is closed, and the arbitrary intersection of closed sets is closed, too. If we see these statements and we are comparing them with the concept of open sets, so if we are beginning with open sets, let us take their complements. We can easily get these characterizations. For example, we know that the complement of the empty set is X , which is always in \mathcal{T} . So, we can say that the empty set is closed. Similarly, if we are finding X^c , that is an empty set, which is open. Therefore, we can also conclude that X is a closed set.

Continue with it; we are taking a finite number of closed sets F_1, F_2, \dots, F_n . If we want to justify that their union is also closed, this is a simple one because if these are closed, $F_1^c, F_2^c, \dots, F_n^c$, are open sets. From the properties of open sets, $F_1^c \cap F_2^c \cap \dots \cap F_n^c$ is also open. Thus, using De Morgan's law, $(F_1 \cup F_2 \cup \dots \cup F_n)^c$ is open. Therefore, we can conclude that $F_1 \cup F_2 \cup \dots \cup F_n$ is a closed set. Thus, the finite union of closed sets is closed.

Coming to the last one, if we are taking a family of sets $\{F_i : i \in I\}$, and we are taking that F_i is closed, for all $i \in I$. Then, in order to show that $\cap\{F_i : i \in I\}$ is a closed set, let us find its complement. By using De Morgan's law, the complement can be written as $\cup\{F_i^c : i \in I\}$. Note that if F_i is closed, then F_i^c is open. Also, we know that an arbitrary union of open sets is open. So, $\cup\{F_i^c : i \in I\}$ is open, and therefore, $\cap\{F_i : i \in I\}$ is a closed set. Thus, we have deduced all three of these properties.

It is interesting to talk about the arbitrary union of closed sets. It is to be noted that an arbitrary union of closed sets may not be closed. For example, if we are taking Euclidean topology on the real line, let us take the intervals of the form $[1/n, 3]$, where n is a natural number. What about $\cup_{n \in \mathbb{N}} [1/n, 3]$? This

union will be nothing but a semi-open interval $(0, 3]$. Note that this interval is not closed in Euclidean topology because if we are looking the complement of this interval, that will be nothing but $(-\infty, 0] \cup (3, \infty)$. There will always be a problem at 0, and because of it, this union cannot be an element of Euclidean topology. Therefore, this is not a closed set.

These are the references.

That's all from today's lecture. Thank you.