Course Name: Essentials of Topology
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Welcome to Lecture 10 on Essentials of Topology.

In the previous lecture, we studied Euclidean topology and right ray topology on $\mathbb{R}$. Continuing from there, in this lecture, we will study two more topologies on $\mathbb{R}$, known as lower limit topology and upper limit topology.

Begin with the concept of lower-limit topology. The construction is similar to the concept of Euclidean topology. Let

$$\mathcal{T} = \{G \subseteq \mathbb{R} : \forall x \in G, \exists a, b \in \mathbb{R} \text{ with } a < b \text{ such that } x \in [a, b) \subseteq G\}.$$

Let us see how this $\mathcal{T}$ is a topology on the set of reals.

Begin with the empty set, which is trivially a member of $\mathcal{T}$. $\mathbb{R}$ will also be a member of $\mathcal{T}$ because for $x \in \mathbb{R}$, $x \in [x, x + 1) \subset \mathbb{R}$. Moving ahead, let us take $G_1, G_2, ..., G_n \in \mathcal{T}$. Note that if any of G_i, $1 \leq i \leq n$, is an empty set. Then $G_1 \cap G_2 \cap ... \cap G_n$ is an empty set, and therefore, $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$. Now, we take all G_i's as non-empty sets, $1 \leq i \leq n$. Also, let us take $x \in G_1 \cap G_2 \cap ... \cap G_n$. Then $x \in G_1$, $x \in G_2$, ..., $x \in G_n$. But at the same time, it is to be noted that $G_1, G_2, .., G_n$, all are from $\mathcal{T}$. Thus there exist $b_1, b_2, ..., b_n \in \mathbb{R}$ such that $x \in [x, b_1) \subseteq G_1$, $x \in [x, b_2) \subseteq G_2$, ..., $x \in [x, b_n) \subseteq G_n$. Now, what we can do that, let us take a real number b, which is a minimum of $b_1, b_2, ..., b_n$. Then, we can conclude that $x \in [x, b) \subseteq [x, b_i) \subseteq G_i$, where $1 \leq i \leq n$. Therefore, $x \in [x, b) \subseteq G_1 \cap G_2 \cap ... \cap G_n$. Thus, $G_1 \cap G_2 \cap ... \cap G_n$ is a member of $\mathcal{T}$.

Moving ahead, let us take a collection $\{G_i : i \in I\}$ of subsets of $\mathbb{R}$, where every G_i is in $\mathcal{T}$. Our motive is to justify that $\cup\{G_i : i \in I\} \in \mathcal{T}$. For which, let $x \in \cup\{G_i : i \in I\}$. Then obviously $x \in G_k$, for some $k \in I$. Because G_k

is a member of $\{G_i : i \in I\}$, therefore $G_k \in \mathcal{T}$. If $G_k \in \mathcal{T}$, there exist real numbers a and b with $a < b$ along with the condition that $x \in [a, b) \subseteq G_k$. But note that $G_k \subseteq \cup\{G_i : i \in I\}$. Thus, $x \in [a, b) \subseteq \cup\{G_i : i \in I\}$. Therefore, $\cup\{G_i : i \in I\} \in \mathcal{T}$. That is, $\mathcal{T}$ is closed under arbitrary union. Therefore, $\mathcal{T}$ is a topology on $\mathbb{R}$, and this topology is known as the lower limit topology, denoted by $\mathcal{T}_l$.

Let us analyze the subsets of $\mathbb{R}$, which are $\mathcal{T}_l$-open or not. Let $a, b \in \mathbb{R}, a < b$. Then

- $(a, b) \in \mathcal{T}_l$ as for all $x \in (a, b), x \in [x, b) \subseteq (a, b)$.

- $(a, \infty) \in \mathcal{T}_l$ as for all $x \in (a, \infty), x \in [x, x + 1) \subset (a, \infty)$.

- $(-\infty, b) \in \mathcal{T}_l$ as for all $x \in (-\infty, b), x \in [x, b) \subset (-\infty, b)$.

- $[a, b) \in \mathcal{T}_l$ as for all $x \in [a, b), x \in [a, b) \subseteq [a, b)$.

Moving ahead, we can see that $(a, b]$ and $[a, b]$ are not members of the lower limit topology. If we want to see it, let us take the example of this semi-open interval $(a, b]$, and try to justify that this is not a member of the lower limit topology. For justification, we are going by contradiction. Note that $b \in (a, b]$. Let us assume that there exist $c, d \in \mathbb{R}$ with c is less than d such that $b \in [c, d) \subseteq (a, b]$. Then we can see that there will be a problem. What is the problem here? Note that $b \in [c, d)$. Therefore, $c \leq b < d$. From here, we can move one more step. That is, $c \leq b < (b + d)/2$, and that is less than d. From here, one thing is clear that $(b + d)/2 \in [c, d)$, but the question is, whether $(b+d)/2 \in (a, b]$. The answer is no because $(b+d)/2$ is strictly greater than b, and that's a contradiction. Therefore, no such real numbers c and d exist. Thus, $(a, b]$ is not a member of the lower limit topology. With the same justification, one can think about $[a, b]$.

Coming to the next one. The question is whether finite sets, the set of natural numbers, the set of integers, the set of rational numbers, and the set of irrational numbers are members of the lower limit topology. The answer is no. Just like the case of Euclidean topology, in the case of finite sets, intervals will contain an infinite number of real numbers. In the case of the set of natural numbers, an interval will contain an infinite number of rationals and irrationals. Same with the case of the set of integers. If we are looking for the

set of rational numbers, there will be an infinite number of irrationals within the interval. For the case of the set of irrational numbers, if we are taking a semi-open interval as a subset of the set of irrational numbers, again, there will be a problem because that interval will contain an infinite number of rational numbers. So, no such sets are members of the lower limit topology.

Similar to the concept of lower limit topology, there is one more concept that is known as upper limit topology. This topology is obtained by choosing a different type of semi-open interval. The semi-open interval, which we have chosen in case of lower limit topology, was closed from the left side. But what are we choosing here? We are choosing a semi-open interval, which is closed from the right side. Specifically, the following is also a topology on $\mathbb{R}$.

$$\mathcal{T} = \{G \subseteq \mathbb{R} : \forall x \in G, \exists a, b \in \mathbb{R} \text{ with } a < b \text{ such that } x \in (a, b] \subseteq G\}.$$

The proof is similar to the proof of the lower limit topology. This topology is known as the upper limit topology, and we will denote it by $\mathcal{T}_u$. Let us have some analysis about this topology too.

Similar to the concept of lower limit topology, all open intervals along with the intervals of the form $(a, b]$ are open sets with respect to the upper limit topology. In this figure, it is clear that $(-4, -1)$ is a member of the upper limit topology. Even the interval $(2, 4]$ is also a member of the upper limit topology.
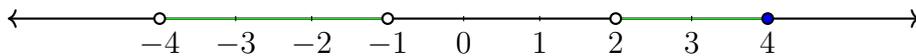

Figure 1: Intervals like $(-4, -1)$ and $(2, 4]$.

Note that the intervals of the form $[a, b]$, or $[a, b)$ cannot be members of the upper limit topology. The question is why. The answer is similar to the case of the lower limit topology; the problem will arise at the left end. Both the intervals contain a, but we cannot construct a semi-open interval, which is desired for the upper limit topology containing a inside these intervals. Therefore, these types of intervals are not members of the upper limit topology. Finally, the same question is here. Whether finite sets, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R} - \mathbb{Q}$ are $\mathcal{T}_u$-open sets. The answer is no, and the justification is similar to what we have discussed in the case of Euclidean topology and lower-limit topology.

Moving to the next, similar to the concept of Euclidean topology, let us take $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{[a,b) : a, b \in \mathbb{R}, a < b\}$. The question is: whether such $\mathcal{T}$ will form a topology. The answer is again negative. For example, if we are taking $G_n = [1/n, 3)$, where n is a natural number. Then note that this $G_n \in \mathcal{T}$. But if we are computing the infinite union of these G_n, it becomes $(0, 3)$, which does not belong to $\mathcal{T}$. Therefore, this is not a topology. Similarly, we can talk when we replace $[a, b)$ with the interval, that is, the semi-open interval $(a, b]$. So, in both cases, $\mathcal{T}$ will not be a topology. Finally, what we have seen is that open interval (a, b), semi-open intervals $[a, b)$ or $(a, b]$, if we are taking any of these with the empty set and set of reals, that will not be a topology on $\mathbb{R}$, but such intervals help us to study the topologies by using the concept of basis, which we will see in the next lectures.

These are the references.

That's all from today's lecture. Thank you.