Course Name: Essentials of Topology
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Week: 01
Lecture: 01

Welcome to the first lecture on Essentials of Topology. In this lecture, we will discuss the introduction of the course from Geometric, Mathematical, and Historical Perspectives. The contents, which we will discuss in this lecture, are: What is topology? Different branches of topology, how topology provides the power of abstraction and a glimpse of the history of topology. Let us discuss what is topology. From a geometrical point of view, topology deals with the specific properties of geometrical shapes that do not change under continuous deformation, such as bending, stretching, twisting, etc. But the question comes, how does topology differ from geometry? The answer is, for example, if we are taking a wire of some fixed length and let us stretch the wire.

So, these two things are geometrical shapes, but geometrically, they have different lengths. Geometrically, they cannot be the same, but we will see in topology that both objects are the same. Moving ahead, let us bend this wire in the form of a curve. So, this curve and this wire are both different in shape. Geometrically, both are different things, but in the sense of topology, both objects are the same. Moving ahead, let us take another example: a geometrical object in the form of a circle. If we stretch this geometrical object and form an ellipse, again, both the objects are different in the sense of geometry, but in the sense of topology, both the objects are the same.

Let us take another interesting example that is based on the concept of continuous deformation of a coffee mug. We deform this coffee mug. Let us deform; that is a continuous deformation, and we are getting different shapes of the coffee mug, and this coffee mug can be deformed into a doughnut. Geometrically, both are different objects, and both cannot be the same, but in the sense of topology, both objects are the same. This example provides a joke on topologists that topologists cannot distinguish between a coffee mug and a doughnut.

Moving ahead, there are different branches of topology. What we are discussing is a subject for discussion under general topology, which is also known as point set topology. But there are different branches, too, such as algebraic topology, geometric topology, differential topology, etc. What exactly does one do under general topology? Actually, this general topology deals with the basic set theoretic definitions and constructions. If we are moving to algebraic topology, actually, it provides a platform for interaction between algebra and topological spaces. As our interest is only about general topology or point set topology, which we have to study in this course. We are not going into detail about other concepts, such as geometric topology, differential topology, or other branches of topology.

Now, let us discuss what is topology from a mathematical point of view and what is the idea behind topology. The idea comes from the concept of continuity of a function. Let us take a function $f : X \to Y$. The question is, what does continuity of f mean?

Let us take an example where we take X and Y, both as set of real numbers. Moving to the example, this is an example taken from GeoGebra. This graph is for the function $0.3x^2 - 4$. The idea is to check the continuity of this function at x equal to 2. It is clear from here that the left-hand limit of the function at 2, and right-hand limit of the function at 2 is -2.8, and the value of the function at 2 is -2.8. But beyond this, the interesting thing is that whenever we are coming to 2, the function values are moving towards -2.8. In another sense, we are discussing something like closeness around 2, or we are talking about the points which are closer to 2, and if the points are coming closer to 2, how the function values are also coming to -2.8. So, what we are getting from here is that if we are saying that x and y are elements of X, that are close together, then the function values $f(x)$ and $f(y)$ are also close together.

So, what do we require for continuity of function? Actually, we need to provide some sense to a notion of closeness for elements in X and for elements in Y. The next question will be how to describe the idea of closeness. One of the simplest ways is why not to use the notion of distance between elements, meaning, if we are taking this as a set X, let us take another set, that is, Y. Take a function from X to Y, two elements of X. For example, this is x, and this is y, and let us take their function values in Y. This is $f(x)$, and this is $f(y)$. So, if we take the distance between x and y, let us take the distance between $f(x)$ and $f(y)$. So, when we are discussing about closeness, the meaning is to assign a real number, $d(x, y)$, for each pair of elements of X, which we are saying that this is the distance between x and y. Meaning is that x and y are close together if the distance between x, y is sufficiently small. So, is it similar to the notion of metric space? Just think about it.

Now, coming to a set X that appears as an unorganized collection of its elements with no further structure. Meaning is that a set is given to us; for example, this is a set. There are different elements in the set, and no other structure is available on this set, something like distance. What have we discussed in the previous case? The question comes that how one can discuss about the idea of closeness between the elements of this set. The answer is that instead of specifying the distance between any two elements x and y of set X, we will give a meaning to which a subset G of X is open. Actually, open sets encode closeness in the following interesting manner.

We say that a set G is open if we are taking any element x in G, then all the elements y in X, that are sufficiently close to x will also satisfy this particular criterion, that y lie in G. Meaning is, if G is an open subset of X, and x is any element of G. Then all the elements which are sufficiently close to x, also will lie in G. In terms of a diagram, for example, if we are taking this as the set X, and we are taking another set that is a subset of X. Let us take that this is G, so there may be elements in the set lying inside G and outside G.

So, for example, this element is x, whenever we are discussing that which elements are closer to x, meaning is that the elements which are lying inside G. So, these are the elements which are closer to x, but the elements which are lying outside G, they are not close to x. This is the idea of closeness described by using the concept of an open set. So, what exactly have we got? We have established, or we have developed some shape; we have seen some shape of X by using the notion of open subsets, and there is no such structure like distance. This specification, along with some additional conditions, lead to the idea of topological spaces.

Whenever we are discussing this particular idea of closeness in the sense of open subsets of X, note that this whole theory depends on sets and subsets. What exactly are we required to study point set topology, the idea about subsets, and even the collection of subsets. It is, therefore, important for persons who want to study topology; they should have some familiarity with the fundamental notions and concepts of set theory. Accordingly, we will begin this course with a summary of concepts related to sets as well as functions. Moving to the next concept, let us see how topology provides the concept of power of abstraction.

We know the Extreme Value Theorem, which states that given a continuous function f from the closed interval $[a, b]$ to set of reals $\mathbb{R}$, f achieves its maximum and minimum in the interval itself. Let us see the graph of a function f defined on the closed interval $[a, b]$. It is clear from this graph that this function achieves its maximum at c and minimum at d. The question is: is it possible to replace the domain of this function, which is a closed interval $[a, b]$ by using some more general mathematical structure? The idea is motivated from the fact that this closed interval $[a, b]$ is a closed and bounded set, and the idea of closed and bounded sets provides the concept of compactness. Therefore, there is a natural question: is it possible that can we replace this closed interval $[a, b]$ by using a compact set? The answer is positive, and this theorem can be generalized in the following manner.

Given a continuous function f from a compact set C to a set of reals $\mathbb{R}$, f achieves its maximum and minimum in C. What have we seen till now? We have made a change on the domain of the function, and we replaced a closed interval $[a, b]$ by using a compact set C. A natural question comes? Is it possible to replace this whole domain of the function by using some more general structure? Again, the answer is positive, and the answer is given by this particular theorem, which states that given a continuous function f from a compact set C to a topological space X, the image of C under f is compact. So what we have seen here is that a continuous function, which was defined from a closed interval $[a, b]$ to the set of reals, can be generalized in more general situations to a continuous function from a compact set C to topological spaces. So, our conclusion is we can expand existing results to more generalized situations, and obviously, this is one of the excellent examples of the idea of the power of abstraction. Even this concept is in line with the quote: Topology is precisely the mathematical discipline that allows the passage from local to global, given by Rene Thom, a French mathematician.

The introduction will not be completed if we are not discussing the history of topology. Let us have a glimpse. The term topology, that is also topologie in German with a little bit of a change in the spelling. This was coined by Listing in 1836. Mathematicians associate the emergence of topology as a distinct field of mathematics after the publication of Poincaré in 1895, entitled Analysis Situs. Hausdorff's 1914 book, written in German, whose English version is Fundamentals of Set Theory, established the foundation of topological spaces and initiated the general study of topology as an abstract mathematics discipline. Beyond this, between the 19th and 20th centuries, a number of mathematicians worked on the development of different branches of topology, which cannot be forgotten. For example, the work of Brouwer, the well-known Brouwer's fixed point theorem in topology, which states that for any continuous function f sending a compact convex set to itself, there exists x_0 in its domain such that $f(x_0) = x_0$. The work of Fr$\acute{e}$chet, even his doctoral work, was towards abstract point set theory or general topology.

During this period, the work of Cantor, who showed that the Cantor set is nowhere dense but has the same cardinality as the set of real numbers, whereas the rational numbers are everywhere dense but countable. The work of Riesz, who studied a preliminary version of the notion of topological spaces called the mathematical continuum.

These are the references which we have used and will continuously use throughout the lectures. The book on Introduction to Topology, Pure and Applied by Adams and Franzosa. The book on Topology by Munkers, and the book on General Topology by Willard.

That's all from this lecture. Thank you.