Engineering Mathematics-II
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Lecture-07
Surface Integral (Part-1)

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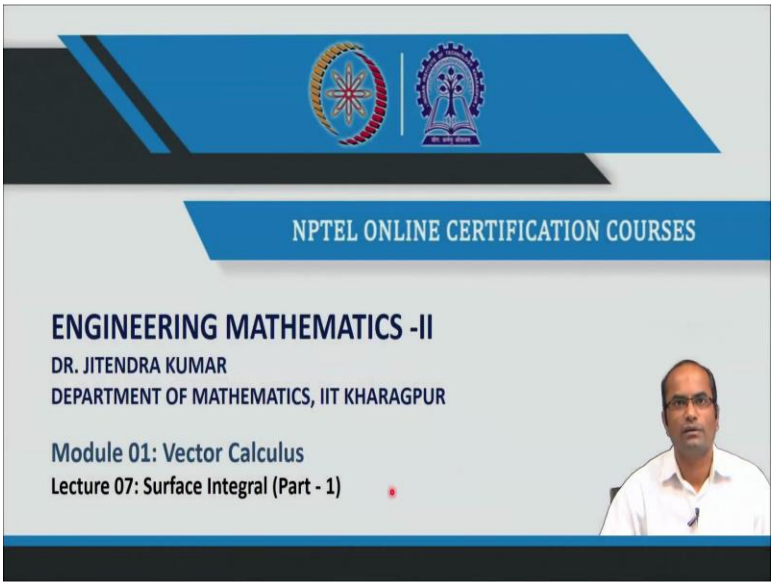

So, welcome back to lectures on Engineering Mathematics II. This is lecture number 7, where we will discuss surface integral and there will be two parts of this lecture, so, this is part one.

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So, here we will be talking about the smooth surfaces and the evaluation of the surface area and that will be the motivation for the surface integral of a scalar function.

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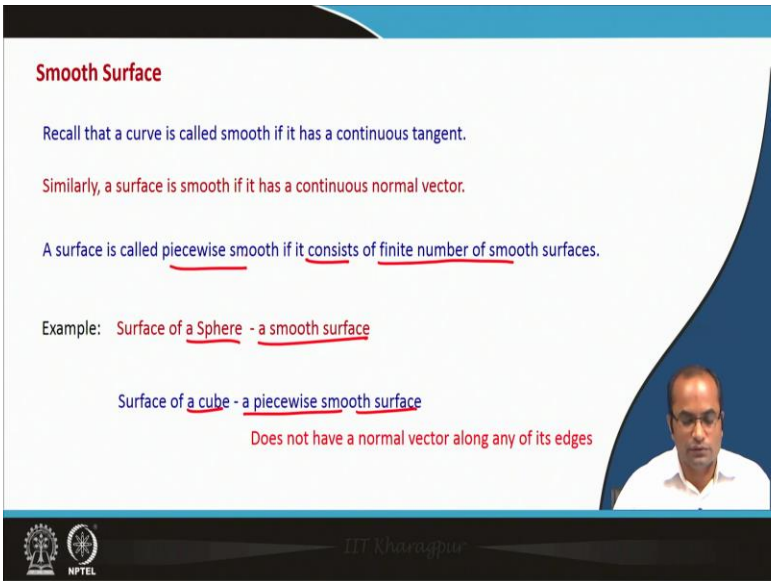

So, what are the smooth surfaces? If we recall that a curve is called smooth if it has a continuous tangent. So, in this case, instead of this continuous tangent we will be talking about the normal to the plane, to the tangent plane. So, similarly, a surface is smooth if it has a continuous normal vector. And a surface is called piecewise smooth, so, for the piecewise smooth we will be talking about that if it consists of finite many number of smooth surfaces.

So, exactly the parallel definition what we have for the smooth curve or the piecewise smooth curve. So, for example, if we consider the surface of a sphere, so this is an example of smooth surface. And if we consider the surface of a cube, this is not smooth, it is piecewise smooth because if we look at the corners of this, not only the corners, but the edges, then along any of its edges, there is no, normal vector. So, that is the reason here we have the smooth surfaces. So, each phase is a smooth surface, but as a whole this cube will not be a smooth surface and we will call it piecewise smooth surface.

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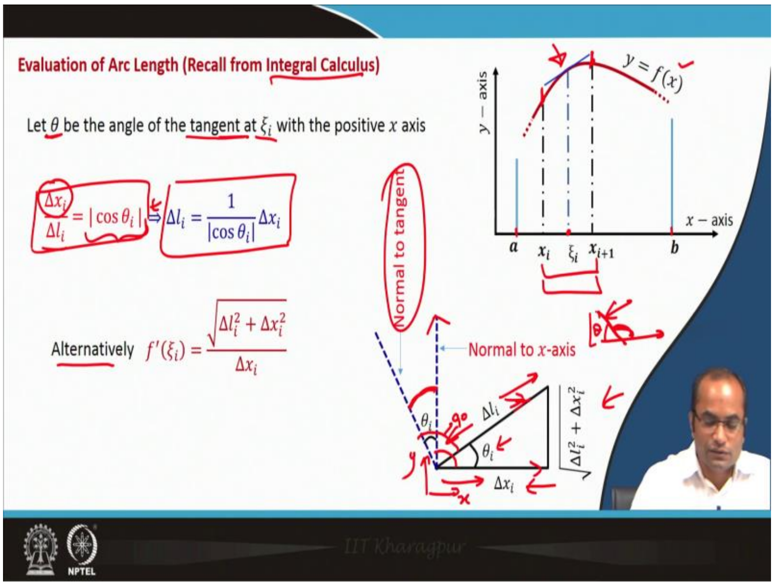

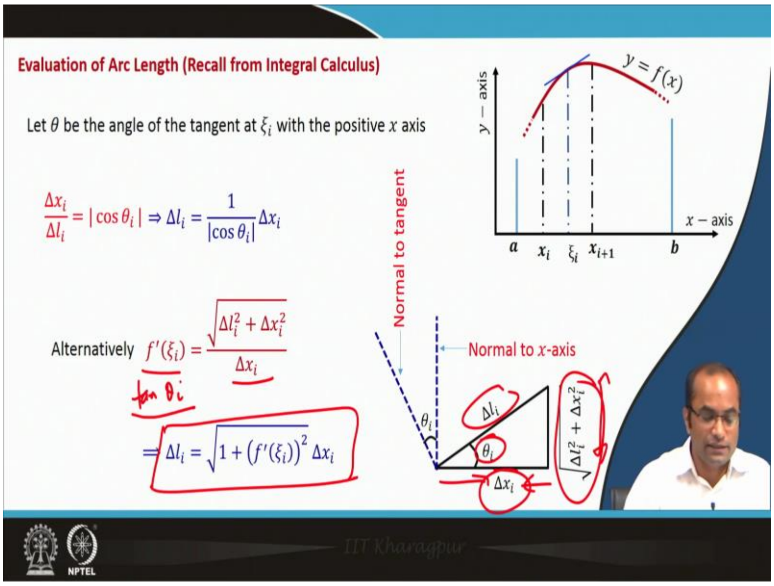

So, first we will recall from the integral calculus how do we evaluate the arc length because that same idea of the arc length will be translated for computing the surface area. So, we assume that this Theta be the angle of the tangent at a point here at the Xi i. So, we have a curve here y is equal to f x, and then which we want to get, for instance, the arc length between this point a and b.

So, what we do usually, we divide this whole arc length into pieces. And suppose this is one piece where our arc length is defined from this point to this point, and there is a point Xi i somewhere between xi and xi plus 1. So, at this Xi i, we have a tangent here and we suppose that this Theta is the angle of the tangent, at this point Xi i with the positive x axis. So, this is the situation we have this positive x axis here, this is x axis, this is y axis.

So, from this x axis we have this angle Theta i which we are denoting. And then this Theta this angle Theta i we can also understand in a different context. So, either we call that this is the angle which this tangent is making, this tangent is making from the positive x axis or we can also say that this is the angle where normal to this, so this is the normal to the tangent at that point and this here, this is normal to this x axis.

So, we have shifted this line by 90 degree and similarly, this line also we have shifted 90 degree to have this normal to the tangent. So, either we say that this theta is the angle of this tangent from the positive x axis or we call that this is the angle between normal to the axis and normal to the tangent. Because this concept here having this angle will be directly translated for the case of surface, because there we are going to consider the normal to the tangent plane.

So, in this case, so this theta is the angle which is from the positive x axis this tangent is making. Then this relation here delta xi over delta li. So, this distance here is delta li. And then we have, sorry this is the Delta li along this tangent at that point and delta xi is the increment here in the x axis we have, which is covering this particular portion of this arc. So, this delta xi, this base and divided by this delta li is equal to the angle cos theta.

So, that is obvious from this figure that cos theta is delta xi over delta li. We have taken the absolute value of cos theta i because there is a reason. If for instance, the direction of this tangent is in this way and then we are taking from the positive axis, so this is greater than 90 and this cos will be negative. But this ratio we do not, we will consider as a negative because this data will be considered and then we have again the same relation.

So, if we take this absolute value of this cos Theta that is fine, we have this relation that cos xi over cos li is equal to absolute value of cos Theta i, and that is valid for whether tangent is making more than 90 degree angle or it is less than 90 degree angle, does not matter. So, we have this relation, what is this relation, that this increment there in delta x i, which is covered by.

Or in other words, that is a projection of this arc length which we are considering here, piece of this arc length which we are considering, it is having the projection on the x axis whose coverage here is delta x i that means, x i plus 1 minus xi. And this ratio so, delta xi divided by this delta li, the length delta li is along the tangent. So, it is natural again to see here that delta li will be always greater than Delta xi.

And this ratio what we are calling here, that is the cos Theta i and its absolute value. So, we have this relation here, that delta li is equal to 1 over cos theta i and delta xi. So, this Delta li is along the tangent. And if we sum all these distances of every pieces, and then this we let finally that this delta x goes to 0. That means, along this tangent will become exactly the arc length.

So, this is the idea of the arc length which we have learned from the integral calculus and that we will apply here to get this arc length. Alternatively, because this figure here says that, we have delta xi this distance and then we have, this is delta li square plus delta x i square and the square root and here we have this delta li.

So, we have this relation that the derivative, the tangent at this xi. So, that means tan of Theta i, the tan of Theta i, this is tan of Theta i, the tangent at this point is equal to, we have this one here delta Li square plus delta x i square, this length and divided by delta xi, so, divided by this base. Then from here we can get this relation that delta li is equal to square root 1 plus f prime square delta xi. So, this is also a familiar expression for this delta li.

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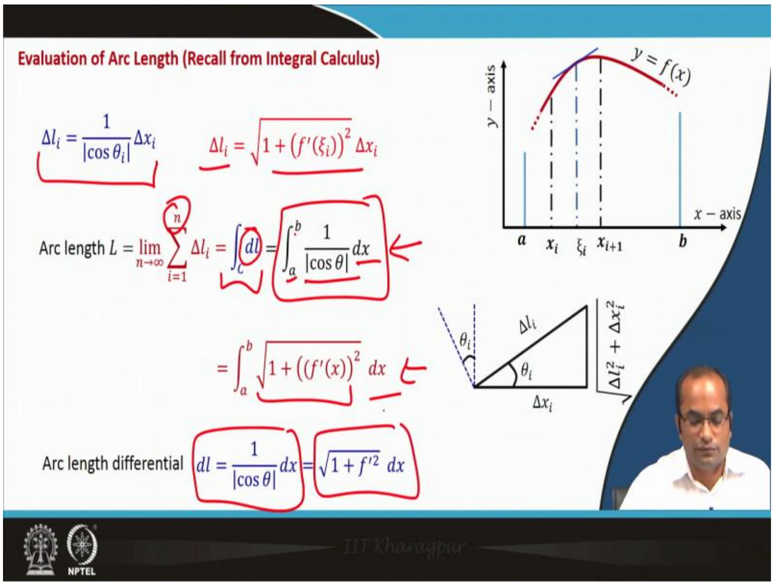

Okay, so, we have these two relations either Delta li can be written in terms of this theta or delta Li can be written in terms of this derivative. In either case, if you want to find the total arc length L. So, we have to sum these Delta Li and take the limit as these pieces goes to infinity or delta x goes to 0. So, and this becomes this arc integral or the line integral which can be used to evaluate the arc length.

So, here this delta L element, this differential element on this arc, we can replace then 1 over cos Theta and dx. So, this will be the integral which can give us this arc length of the whole curve because we are integrating over x from a to b. So, either this formula is applicable or instead of replacing 1 over cos Theta dx we can also replace by the square root 1 plus f prime square dx. So, we have the two formula, either this can be used or this can be used. So, the arc length differential here dl can be either expressed by this or it can be expressed in terms of f prime.

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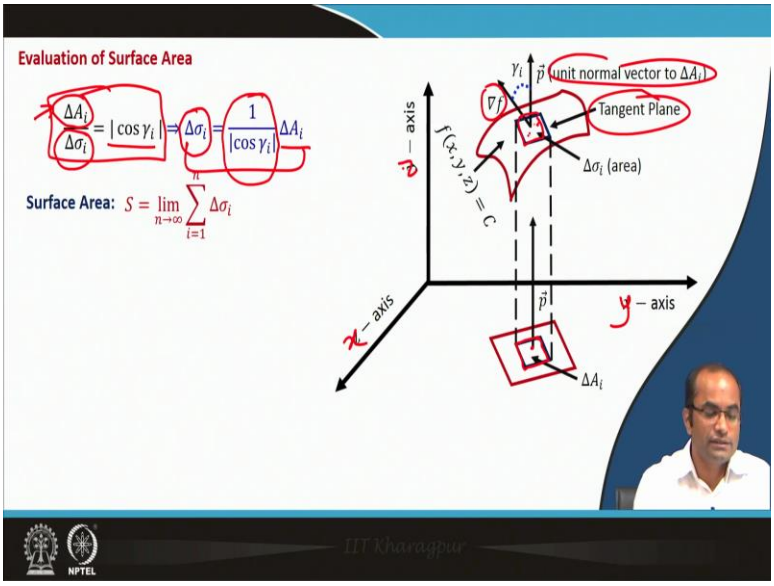

So, for getting the evaluation of the surface area, we will just try to translate from the idea of the arc length. So, this relation, there the same relation holds only now the difference is that we have this surface and we have fixed a point here and this is the tangent plane, this is the tangent plane here at that point. And the normal to this tangent plane will be given by this delta f and the gradient f.

And then this another vector we have considered that is this P here, that is normal to this delta i which is on the x y plane or x z plane in this figure for instance. But it can be on any of the coordinate planes, the perpendicular to the coordinate planes. So, the idea here is, so, we can here take x and then this is y axis and then this is z axis. So, the idea here that we will project this portion of this surface, so, the surface will be divided again into small pieces.

And suppose this one piece of the surface is projected on the xy plane in this case, then we have this projection of that small surface. And this P is the perpendicular, the unit perpendicular, unit vector to this projected plane. So, in this case this P for example, is going to be just the k in the direction of z axis.

So, here this relation similar to what we have discussed just before, a similar relation holds, that the area of this small portion which is projected on the plane, one of the coordinate planes, divided by the area of this tangent plane, of this small tangent plane, is equal to given by this cos and this gamma i, were gamma i is the angle between normal to the tangent plane and normal to the projected plane.

So, there it was Theta which was normal to this x axis and again the normal to the tangent. So, here we have the same translation what we have just discussed before. So, in that case this relation can be given as that this area on the tangent plane will be just 1 over cos gamma i, this multiple of the area which is projected on the xy plane.

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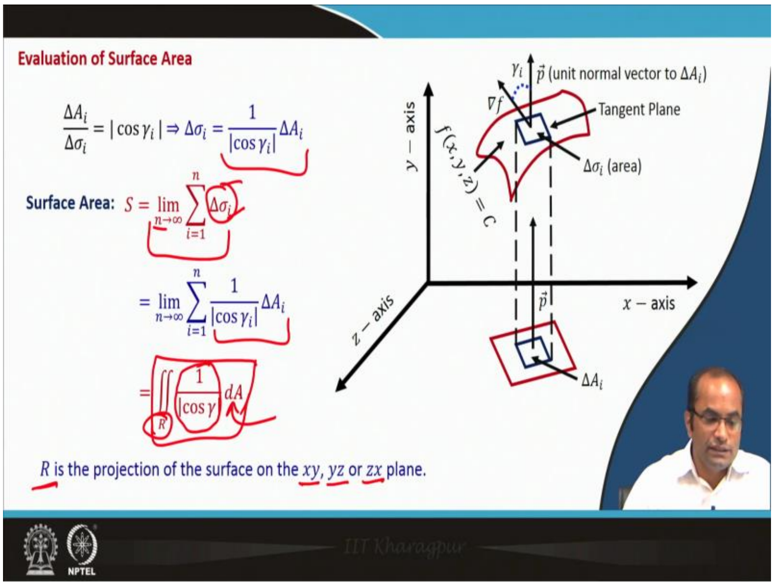

So, now, the surface area we can evaluate easily, again the same idea that we will sum here all these portion of the surface and then we will take this n approaches to infinity. So, at present here this is on the tangent plane, but when we take n approaches to infinity, this will eventually form the surface area. So, we have now replaced this element here delta sigma i by this given element here.

And then this is the integral which can be evaluated to get for instance the surface area. And this r is the projection of this surface on the xy, yz, or zx plane. So, as I said so, we will have a surface we will project to one of the coordinate planes whichever is convenient, and then the surface integral can be evaluated or the surface area can be evaluated by this double integral, because now we are integrating over the plane, such integrals we have already discussed in the integral calculus.

And this is the projected area on one of these planes and this is the additional factor which will cover exactly the relation that the surface area and now, we have in the plane, corresponding area in the plane because there is a relation. Always the surface area of a curve, of a surface is going to be more or equal than the projected one in one of the coordinate planes. So, that factor will take care that part and then we are simply integrating over the plane, double integral which was already discussed in the integral calculus.

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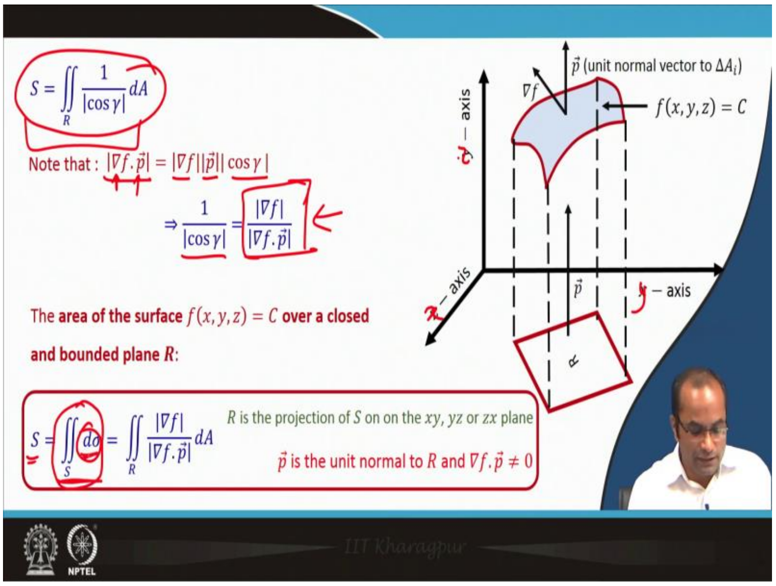

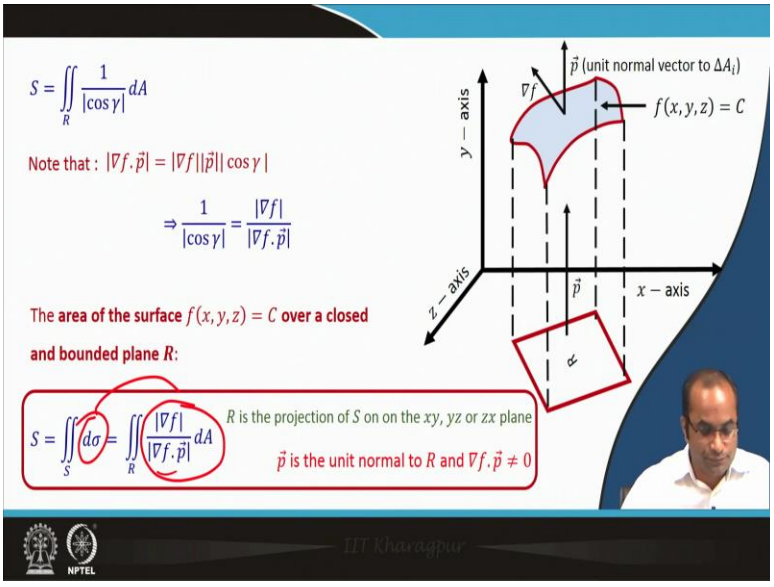

So, we have this surface to get the surface area, we will get another convenient formula for this evaluation. So, note that if we have this gradient of f which is perpendicular to f and then this p unit normal vector to one of the coordinate planes. So, again this is x and y and z. So, here we have this relation that gradient f modulus, gradient p and then the angle between the two which we have already discussed this gamma.

So, 1 over this cos gamma can be written in this in this form. So, here we do not have to compute this gamma n. Because in this form the surface integral, the evaluation is certainly difficult. So, we are forming a new formula where we can simply use to get the surface area so, that is the relation here. And then we have to replace that with this dell f over dell f dot p. So, in this case, we have finally this formula for computation of the surface area.

This is your the element on the surface which, so, here we have the surface integral we are we have a differential element here on surface, and then we are integrating on surface. So, to evaluate this we will simply evaluate this double integral and this r is going to be the projected area on the plane, the projection of this surface to one of its plane here.

So, given f, we can compute this, so, the given surface we can compute the gradient f and the absolute value and here this dot product, we will divide with the absolute value and then we will integrate over r to get the surface area. So, this relation here, the element d sigma is actually directly related to this element on the projected plane dA and this is the additional factor which will cover the difference between the two areas.

Well, so, this is valid of course, that this the gradient f and p, this dot product should not be equal to 0. So, they should not be perpendicular to each other, in that case the projection will be also a problem, because the projection is going to be lying at some point where this is happening. So, we will avoid that the gradient f and dot p should not be 0 they should not be perpendicular to each other, otherwise this formula will not be applicable.

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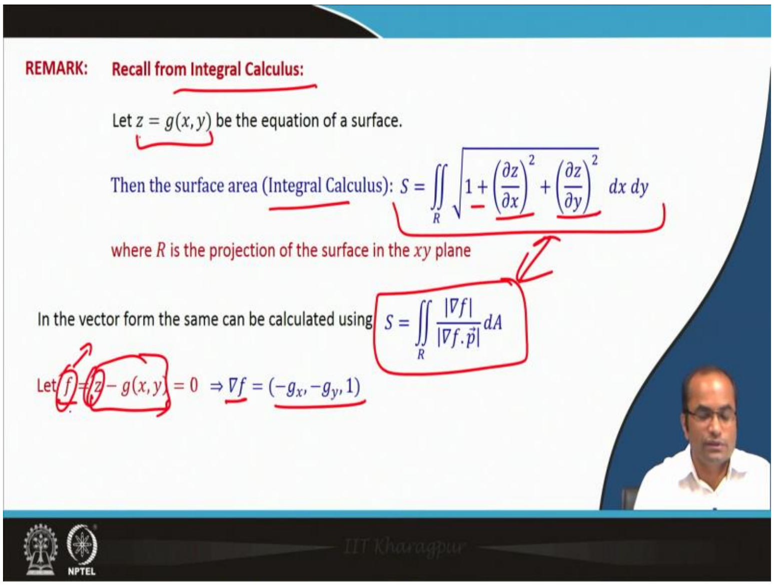

So, just a quick remark that if you recall from the integral calculus when we used to take z is equal to g x, y, this form of the surface, then the surface area in the integral calculus we have discussed can be evaluated by this formula where square root 1 plus zx and zy whole square dx dy was used. And now, we have the another one but we will see that these are basically the same. So, in the vector context what we can, we got already this formula for the evaluation of the surface.

And this can be seen that they are actually the same. So, if we consider this f as z minus this g x, y, because the surface was z is equal to g x, y. So, we have just defined this function f equal to 0, it is like level surface of this f will be exactly this given surface. And then grad f we can compute which is the partial derivative of this with respect to x that is gx minus gy and then with respect to z, so we will get 1.

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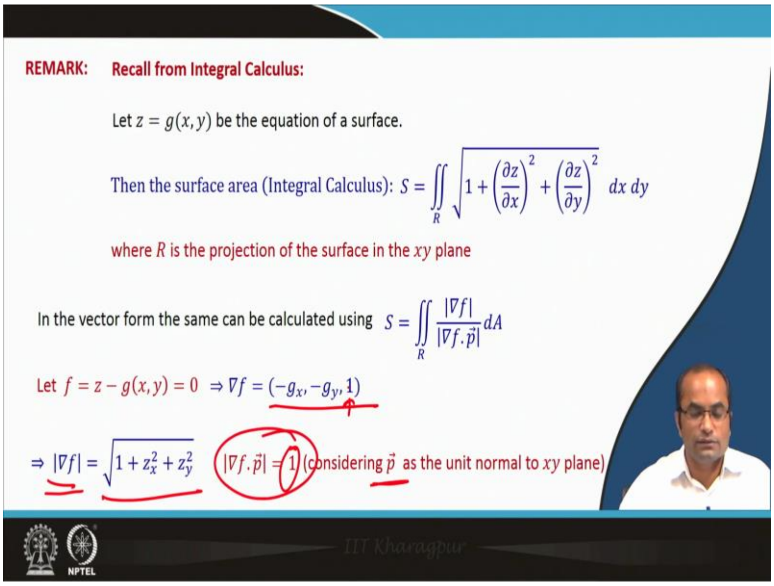

And then this, the absolute value, we can get this square root this g x square g y square, so g is just the z, so we have written form of z. And this is going to be 1 because considering this p was the unit normal vector, and this grad f here we have minus gx minus gy 1. So, this p was the unit vector to this xy plane for instance in this situation, then that is only k. So, only we will have 1 there. So, these two formulas are same, they are not different.

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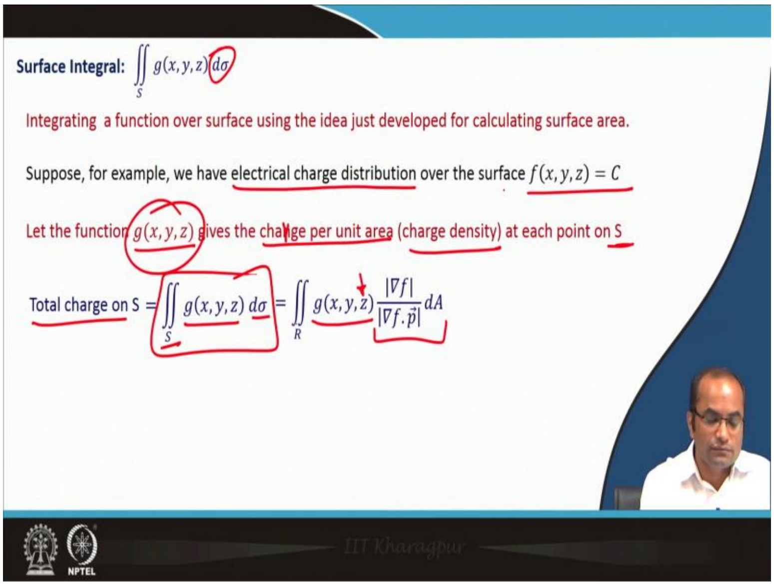

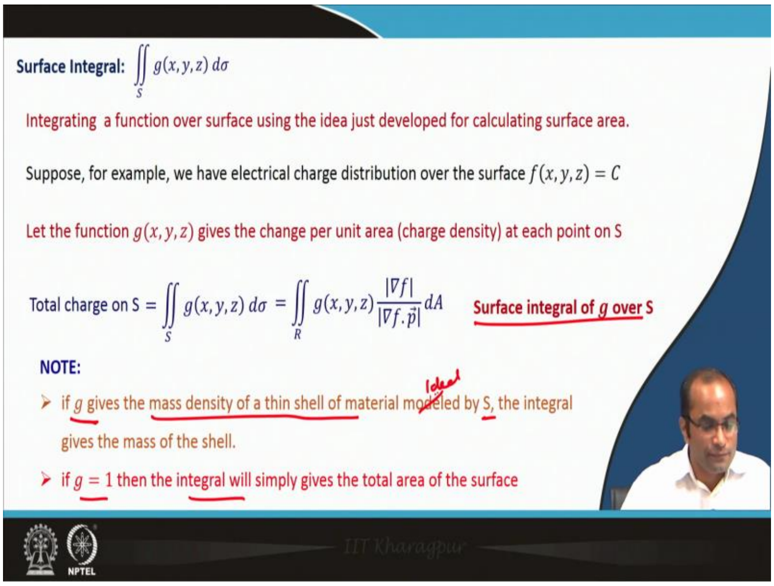

So, just extension of this, instead of just getting surface area where this function was taken as, the integrand was taken 1. So, if we have some other integrand, for instance g as a function of x, y, z some scalar function is sitting there. We can integrate this g over s using the same idea, and the idea is exactly what we have this differential element will be replaced by the differential element on one of the coordinate planes.

So, suppose and just to give you some idea that what are the uses of such integral. Suppose, we have this electrical charge distribution over a surface here f x, y, z equal to C. And in that case, we assume that g x, y, z is the change per unit area, it is the charge per unit area. So, this is the charge per unit area or we call this charge density at each point of the surface S. So, to get this total charge on S, we have to integrate this given density g x, y, z over the surface because we are interested to get the total charge on the surface.

So, at each point we have to basically add up this charge, that means this integral, we have to integrate this. So, we have the surface integral then this is one of the applications where we use this. So, g x, y, z, this density has to be integrated over the surface area and the idea which we have developed. So, g x, y, z will remain as it is the only thing if we are projecting on the xy plane then the z has to be replaced from the surface using the surface.

Or similarly, the other variable if we are projecting on yz, then there should not be x there, that will be replaced from the equation of the surface and then this was the differential element. So, we will integrate this over again projected plane. So, we have the surface integral, this is called the surface integral of g over S.

And just to note that we have more applications for example, if this g is the mass density of a thin shell of material which is moulded by this S here, which is moulded to give this shape S. Then the integral, this integral if this is the mass density, if g is the mass density then this above integral will give us the total mass of that shell. If this g is equal to 1 then naturally we have seen that this formula will reduce to the calculation of the surface area.

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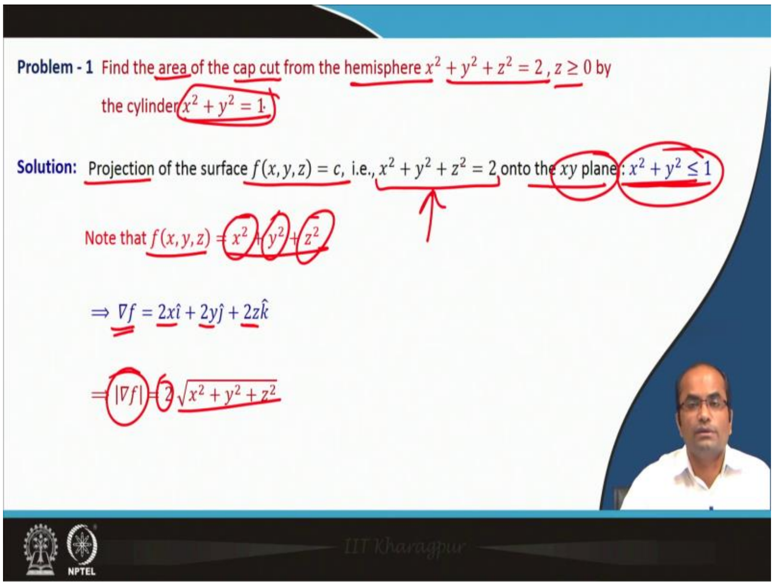

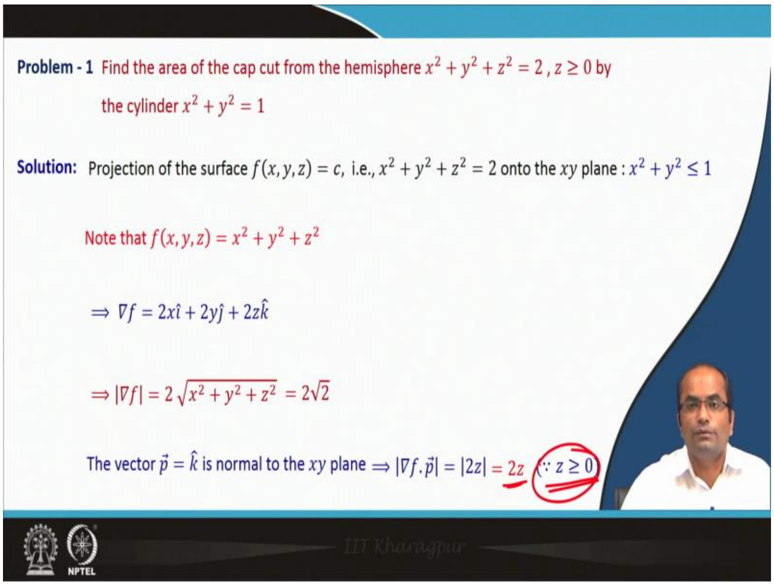

So, having this we can now discuss some of the examples. So, there will be two examples, one we will discuss to find the area of a surface and the other one where we will integrate some scalar function. So, to find this area of a cap cut from the hemisphere, so, we are talking about the hemisphere, this x square plus y square plus z square equal to 2. And since it is a hemisphere we are taking z in the positive access.

So, by this cylinder, so there is a hemisphere which is cut by a cylinder and cylinder the radius is 1. So, we are interested to get that surface area which is cut by the cylinder x squared plus y square is equal to 1 the axis of the cylinder is exactly the z axis. So, the cylinder will cut some portion of this hemisphere and we want to find what is the area of the hemisphere. So, many such examples we can get with the help of the integrals which we have developed.

So, first we will be discussing about the what is the projection of the surface f x, y, z equal to C, that is the projection of this hemisphere on the xy plane. So, let us discuss here, because that is the more suitable where we will not end up with dividing by 0 which was discussed just before that we have to make sure that the tangent, sorry the normal and the normal to these one of the planes they should never be equal to 0.

So, for instance in this case, since the cylinder is along the z axis, and if we project that portion of this hemisphere on the xy plane, so, there will never be the situation which we will just observe now. So, if we project that portion, naturally that projection will be the disk only because that circular cylinder was used to cut this hemisphere. So, as a projection on this xy plane, we will get just the circular this x square plus y square and less than equal to 1.

So, we have the r now, which is the projection on the xy plane. So, now, you note that is f x, y, z, our surface is x square plus y square plus z square so, we can naturally get the gradient f. So, gradient f will be 2x and then 2y and then 2z here and we can get the absolute value of this so, 4x square 4y square 4z square and the square root and this 4 can be taken outside 2. Then we have square root x square plus y square plus z square.

And the value from the surface here x square plus y square plus z square is 2. So, this is nothing but 2 square root 2. So, the vector this p, which is on the direction normal to this x y plane. So we know that the k, this vector is normal to the xy plane. So, we have this p also and now we can use in that formula where we need to compute this one, the grad f dot this vector p and in this case, so, the grad f is here, and then if we do this dot product with the K, naturally this 2z will remain. So we have this 2z with the absolute value. So, this vector is computed and we have since z greater than equal to 0. So, we can simply have here 2z, no absolute value now.

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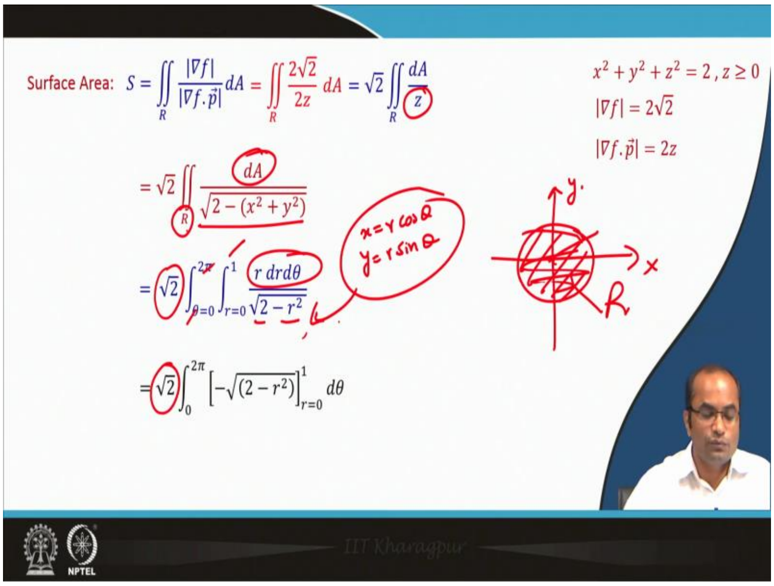

So, the surface area, this integral is to be computed over this R, the projected, which is the circular disk, and we have computed already this Del grad f dot p which is 2z, which we have computed and then this grad f, the absolute value was coming as 2 square root 2. So, we have everything available now here to compute this integral we have 2 square root 2z and then we have 2z there and it has to be computed over dA.

So, since we have projected on this xy plane, there should not be z there, so z must be replaced from the equation of the surface that is x square plus y square plus z square is equal to 2. So, this z we can replace from there and z is positive so we can take the positive square root of this. So, from here what we will get that z is 2 minus x square and minus y square so that can be used here now.

So, we have this z and that will be replaced by square root 2 minus x square plus y square. And we are integrating over this R and R is the circular disk on the xy plane. So, now since we have the circular disk on the xy plane, it is better to use the polar coordinates to compute this integral over this R here. And that means the square root 2 is sitting already outside, the d, the differential element here that will be r dr d theta for the polar coordinate.

The square root 2 minus, so we have x is equal to r cos theta, and we have y is equal to r sin theta. So, from here we will get the 2 minus r square because x square plus y square will be r square. So, we have here this integral and then square root 2, so the limits are clear. So, the limit for r will go from 0 to 1 and for theta 0 to 2 pi. So, that will cover the whole disk. So, now here we have to integrate this, so we have r square root 2 minus r square.

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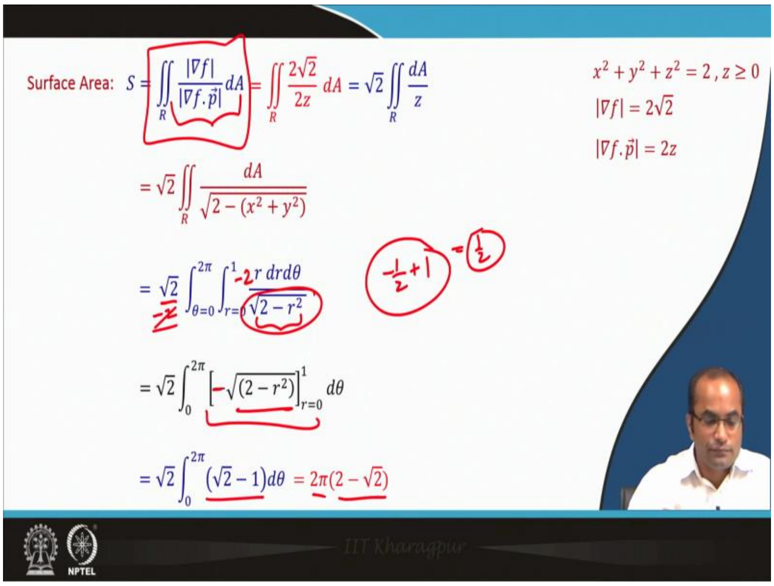

So, the integral of the differentiation of this we need there, so minus 2, minus 2 and then we can divide by 2 minus 2 there to compensate this. So, the integral is sitting there and then we can differentiate this or integrate this so we have power minus 2. And that will be added, so, minus 1 by 2 and then plus 1. So, it will be 1 by 2 there and then we will divide by 1 by 2. So, this will be settled here. So, we have minus sign, and then a square root 2 minus r square, r is 0 to 1. So, we can integrate this and over the 0 to 2 pi.

So, this was just a constant of course, so, we have 2 pi and 2 minus square root 2. So, we have evaluated the surface area which was cut by the cylinder. And we have seen that the surface area can be computed with just the usual double integral, but this differential element here has to be considered which will take care the difference between this curvature part and then we have, we are integrating over the plane.

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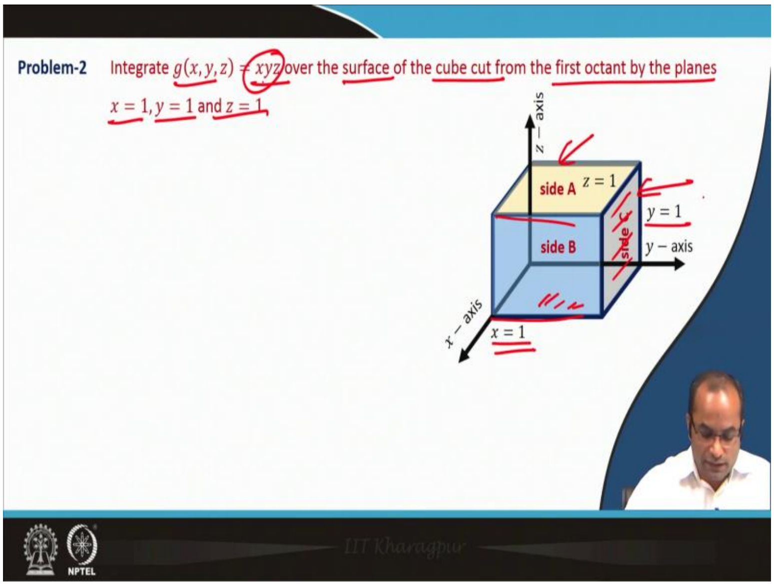

Okay, so we will go through one more example where we will integrate here x y z. So, we are not just computing the area, but we are integrating this function x y z over a surface of the cube, which is sitting in the first quadrant. So, we have a cube in the first quadrant and bounded by this x is equal to 1, y equal to 1 and z equal to 1 plane. So, we have this situation here in the quadrant x y z axis. So, there are six phases here.

So we named it, this is the side A where z is 1, so along z axis, we have this side B, where x is equal to 1 and we have y is equal to 1, this is another side. And there are three more sides which are sitting on these axis. And the interesting part here that we are integrating the x y z, and along all these three, along those three planes which are touching the axis. So, either x is 0, y is 0 or z is 0, so all those three surfaces, the integrand is going to be 0.

So, finally, we will end up with integrating only these ABC, the three surfaces, where for instance, your x is 1, x is constant and yz will vary of course. And similarly, in this surface, the top one the z is fixed and x and y will vary, on this surface your y is fixed and this x and z will vary.

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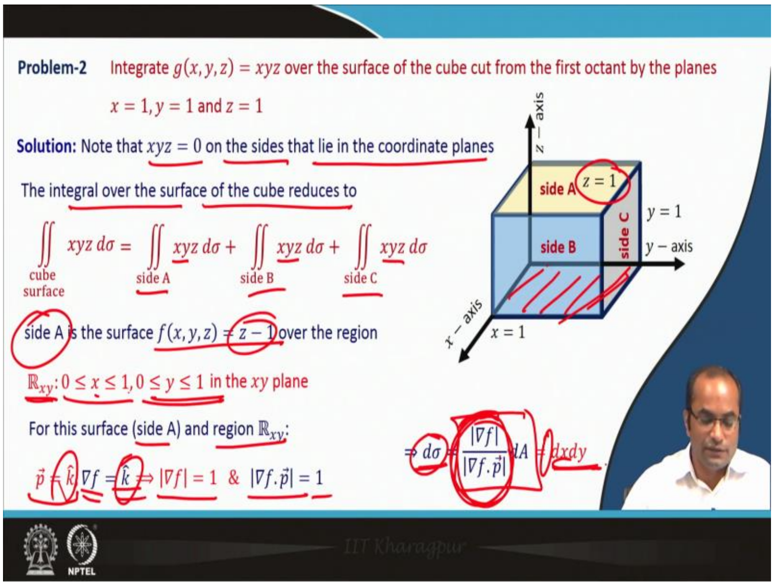

So, this is 0 as discussed, on the sides that lie on this coordinate planes. So, we have the integral over the surface of this cubic, of this cube reduces to the three surfaces. So, side A, side B and this side C, because at these surfaces our integrand will survive, the other three this will become 0. So, that integral will be 0. So, let us consider the surface A here, the surface A is z is equal to 1, this is the surface here z is equal to 1 that plane. And if we project the surface, so, here the surface is also plane and we are projecting naturally over the plane, so, we will have the same area. So, which we will also notice here while calculating this.

So, if we project this on the xy plane, we will have again this square which x will vary from 0 to 1, y will also vary 0 to 1, so that will be the projection of that on this xy plane. And for this surface, the side A and the region, we have the p is k because we have on the xy plane projection, so the perpendicular to this xy plane is is k. And then the gradient f if we compute since f is z minus y, so, we have this dell f again will be also k.

So, only this because z is there, so, when partial derivative of f with respect to z will give 1, so that is the k. And we can compute this grad f, the absolute value, that is 1 and also this grad f dot p that will be also 1. So, here things are much more simplified and this differential element on the surface, surface is a plane which can be computed by this expression. And what we will observe because this is 1 and we have dx dy.

So, again, what you can get this because there is no difference so you have a plane and projecting exactly on the plane, the plane is parallel to the xy plane and we have projected on this plane. So, there will be no difference in the surface area or because there is no curvature there, it is exactly parallel plane to this xy plane. So, this element is going to be 1 and this d sigma where we are integrating over this surface is equal to the integrating over the plane. So, there is no extra element in front of this dx dy differential.

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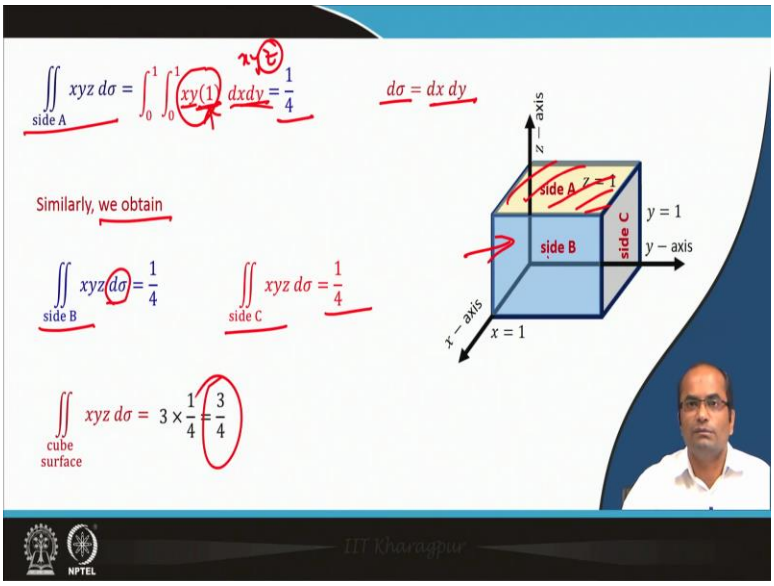

So, now we can integrate easily, so over the side A we have where d sigma is simply dx dy, and the z is 1, so because the integrand is, the integrand was x y z and the z on this surface here, we are integrating on this side A, the z is 1. So, the z is substituted as 1 and we have to integrate just x y over this dx dy. So, this integral is 1 by 4 and the similar calculation we can do for side B and the side C.

So, similarly we obtain for the side B when we project on this yz plane. So, we have here dy dz and similar situation. So, x will be fixed there, but we have then yz, so 0 to 1, 0 to 1 we integrate, we will have 1 by 4. Similarly, for the side C it will be projected on this xz plane, and then we can get again the same value. So, if we add the 3, we will get this 3 by 4. So, that is the value of that integral when we integrate over this cubic surface. So, the integral was much simpler in this case because the differential element on the surface was equal to the differential element on the coordinate axis, on the coordinate planes.

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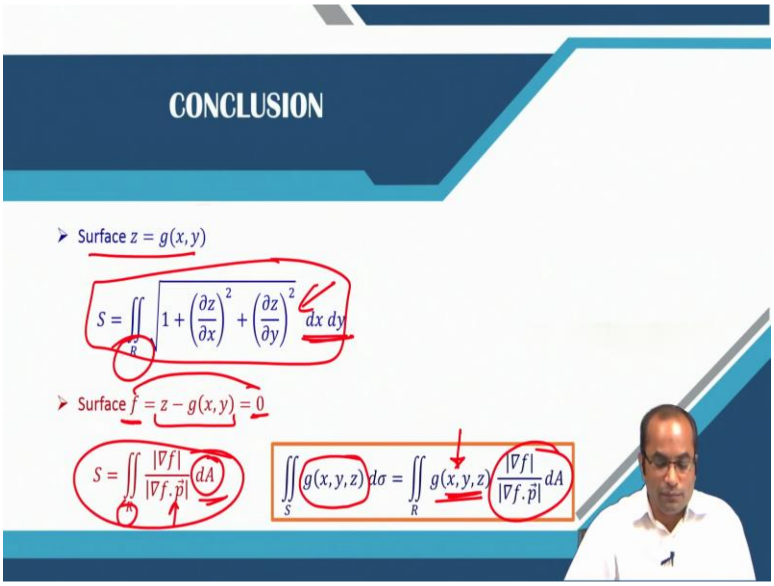

So, to conclude, here we have the surface we have considered z equal to this g x, y. And then this is the familiar formula which we have already used in, for example, the integral calculus. But in the context of the vector we have seen that, if we take this as a level surface this z is equal to g x, y, this surface if we treat as the level surface of this f equal to 0. So, instead of this formula, we can use S equal to this grad f divided by this grad f and the dot product with p and then we can integrate over any of the projected coordinate planes.

So, this was a very particular case when the projection was done on the xy plane, but here we have the more general situation that we can project to any of the coordinate planes accordingly this p either has to be i, j or k depending on the projection. Also we have seen that not only the surface area, we can integrate a function also a scalar function, again the differential element will be replaced.

So, there is no difference instead of this one now this g x, y, z will be used and according to where we are projecting this one of these x y z will be replaced from the from the equation of the surface. So, this was about the surface integral. In the next lecture, we will consider not only that having here a scalar function that what will happen if we have the vector field. So, that will be the discussion of the next lecture. So, that is all for this and thank you very much for your attention.