Engineering Mathematics - I

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Lecture – 53
Exact Differential Equations

So, welcome back and this is lecture number 53. And today we will continue our discussion again on differential equations of order 1 and degree 1.

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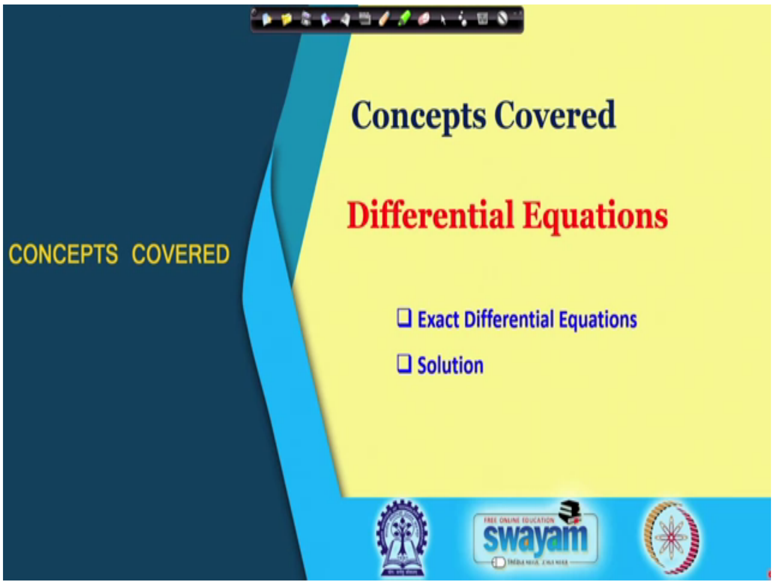

Today's topic will be the exact differential equations, so we will mainly look for the introduction to these exact differential equations and also how to find the solutions of exact differential equations.

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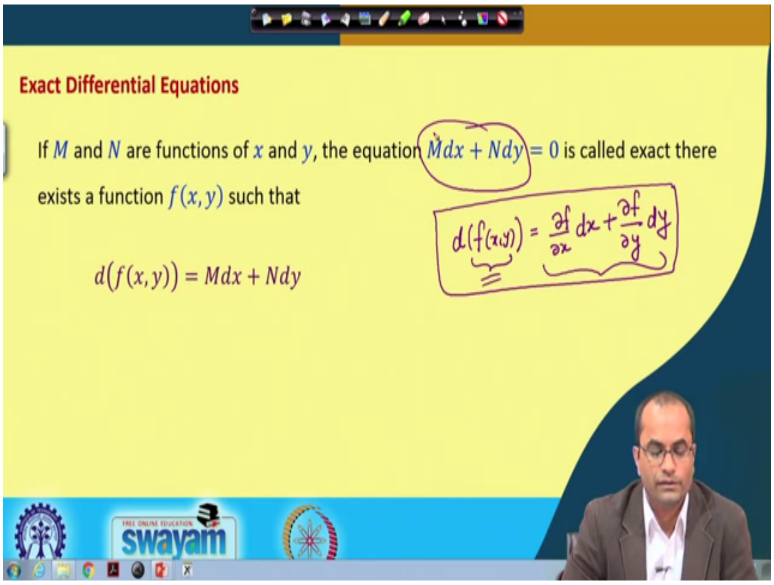

So, what are the exact differential equations? If this M and N are the functions of x and y then this equation Mdx plus Ndy is called exact. If there exist a function this f of x y such that the differential of this function d f x y is equal to exactly this part of the differential equation the left-hand side Mdx plus Ndy. So, if there exists a function this f x y the function of two variable whose differential is exactly this Mdx plus Ndy.

Just to recall what was the differential of the function f x; so, the differential we have discussed already in calculus of two variables. So, the differential of a function of this two variable f x was defined as the partial derivative of this f with respect to x, and this differential of x plus del f over del y and dy. This was the definition of the differential of a function of two variables which we have studied already in calculus of functions of two variables.

So that means, now what we are looking for we are looking for a function f whose differential is exactly this differential equation or rather the left-hand side of this differential equation and. Then we call this differential equation as the exact differential, because this that the differential equation is exactly the differential of some function, and once we find such a function the integration is very easy because we have differential of f here the differential of this f is equal to 0 and then the solution will be this the f is equal to constant.

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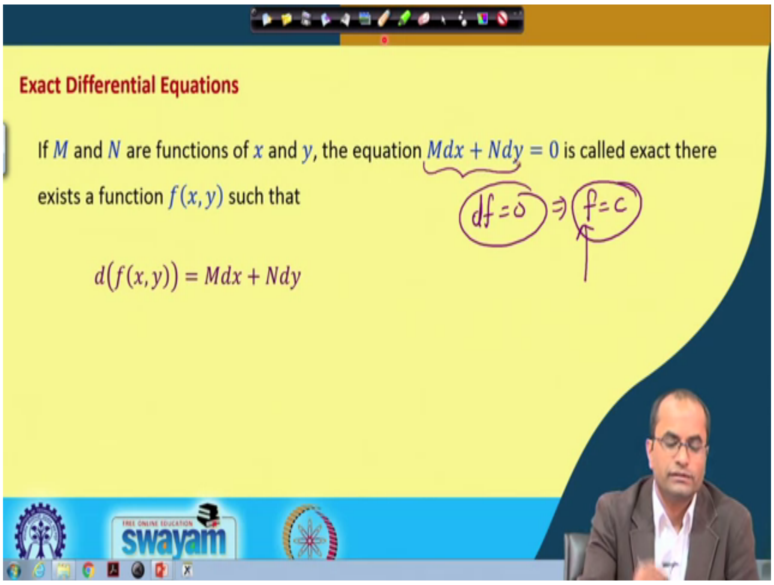

So, one the main job is to find this function f whose differential is exactly this given differential equation.

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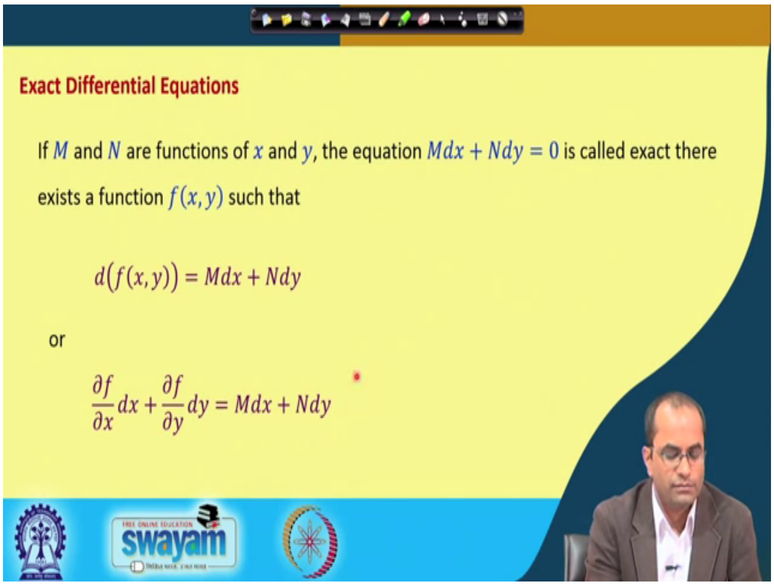

So, either we find this f whose differential is this or in other words as we have discussed already that this differential is defined as the partial derivative with respect to x and dx partial derivative with respect to y dy that must be equal to this Mdx plus Ndy.

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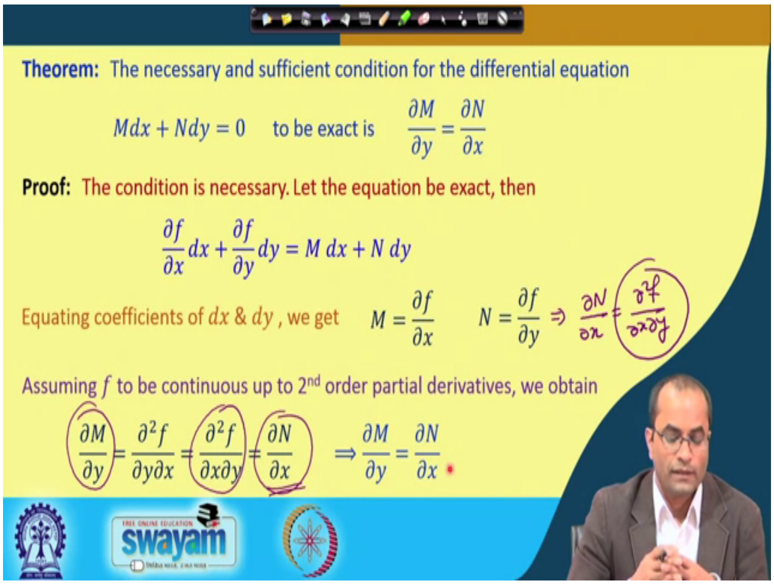

So, there is a nice result here about the necessary and sufficient condition for the differential equation to be exact, and the necessary and sufficient condition for a differential equation here Mdx plus Ndy to be exact is exactly this one here that del M over del y. So, the partial derivative of this function M should be equal to the partial derivative of this function N here. And then not only the sufficient, but this is also the necessary. So, we have both ways here that this condition is also necessary for the exactness of a differential equation and this condition is also sufficient for exactness of a differential equation.

So, since this is a very important result we will also go through the proof of it. So, let us assume that this condition is necessary meaning that if the given differential equation is exact then del M over del y should be equal to del N over del x. So, let the equation be exact, so we assume that the equation is exact and then we will prove that this condition holds.

So, here if this equation is exact then we will have a function f a function of two variable whose differential will be this Mdx plus Ndy as per the definition of the exact differential equations. So, if the differential equation is exact then we must have this del f over del x into dx plus del f over del y into dy is equal to this given differential equation. And equating now the coefficients of this dx and dy, so we have here this dx and the right-hand side also we have dx and then we have this dy here and we also have the dy there.

So, we can equate because this must be equal now. So, del f over del x must be equal to this M and del f over del y must be equal to N. So, equating these coefficients now of tell dx and dy, what we are getting here? The M is equal to del f over del x and N is equal to del f over del y.

So, now we need to assume that this f to be continuous up to second order partial derivatives because we want to assume the equality of the second order derivatives and for that this is sufficient to assume that f is continuous up to second order derivative. So, having, so now, so we get this partial derivative of M with respect to y that means, here also will be differentiating here with respect to y. So, del 2 f over del y del x that is a partial derivative second order partial derivative of f with respect to y and with respect to x.

Assuming this continuity of the second order partial derivatives what we can also reverse the order of this differentiation meaning that this is equal to del 2 f over del x del y. So, here having this del f del 2 f del x del y we can now from here find out that this is nothing, but del N over del x because this del N over del x from here del N over del x will be del 2 f over del x del y and this del 2 f del x del y, so we have replaced here with this del N over del x. So, we got already this condition that del M over del y must be equal to del N over del x. And this is the necessary condition because we have a started with that the given equation is exact and then we got that this mod condition must hold.

And now we will look the other way around, that if this condition holds then we will show that the equation is exact and that we will complete the proof of this necessary and sufficient condition.

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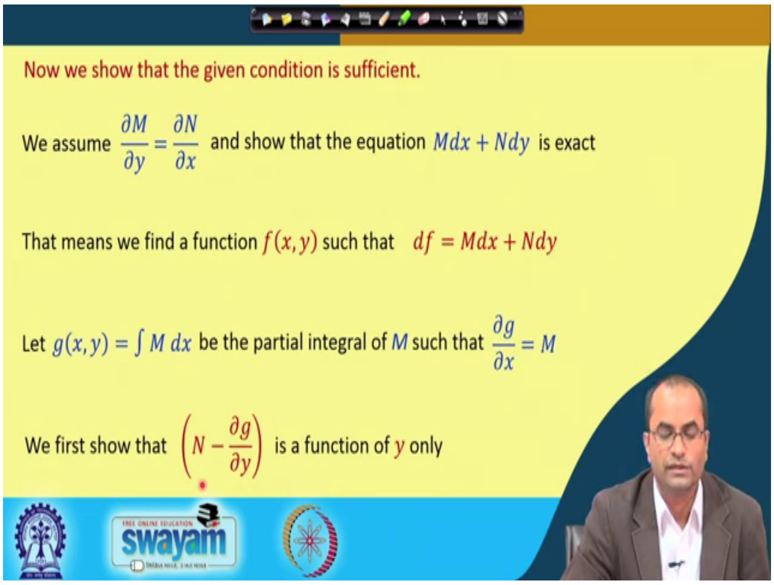

So, here we take now that the given condition is sufficient meaning that we assume that we have this condition or this condition holds, and then we will show that this Mdx plus Ndy is exact. So, to show that this is exact we have to basically get a function f whose differential is exactly this Mdx plus Ndy so that means, we need to find a function here f x y such that this differential of this function is equal to Mdx plus Ndy.

So, for the construction of this such a function. So, this is a little theoretical now. So, you will take a function this g x y as the integral. So, partial integral of this Mdx. So, this is the construction process of this function f. So, here we take this function g as the integral of this M the given M here such that. So, what this integral means here? That the derivative of this partial derivative of this g with respect to x is M because we have just the integral here with respect to x, and we take the differentiation here. So, the partial derivative of g with respect to x from this relation itself you will get this M.

So, having such a g now what we will show first before we exactly construct this f whose differential is M x plus Ndy. We will show that this function here N minus del g over del y. So, N is the given function here in the differential equation and minus this constructed function and its derivative with respect to y, and if we take this difference here this function is a function of y alone this we want to show now before the construction of the function f.

So, to show this meaning we have to just get the partial derivative with respect to x and if it comes to be 0 that means, this is free from x and it is a function of y alone.

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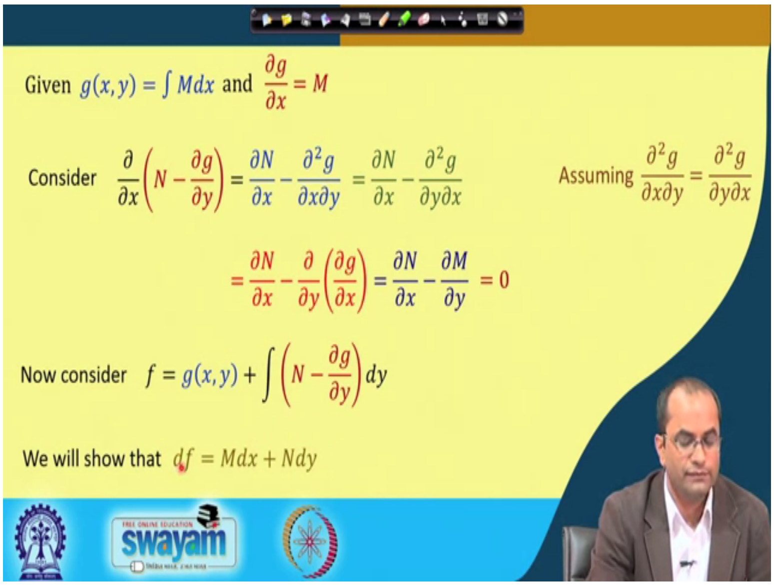

So, these are the given now the g is integral of M del g over del x is M and we want to consider that this partial derivative of this function N minus del g over del y is 0 that we will show now. And that we will tell us that this is a function of y alone. There is no x term in this N minus del g over del y because its partial derivative if we can show to 0.

So, here now the this is partial derivative of N with respect to x and we have the second order derivative of g with respect to x and with respect to y. So, assuming again this equality of the second order mixed derivatives, so we now replace this with del v over del x and here we will take del over del y and del g over del x now. So, we just rewrite this in this form del N over del x and here del over del y of this del g over del x, and this del g over del x is given as M here by construction. So, this is nothing but M.

So, this is del N over del x and minus this del M over del y. This is the given condition this is what we have assumed that this condition holds and we will show that the equation is an exact differential equation. So, here this was the condition that del N over del x is equal to del M over del y. So, with that condition we have seen now, if this condition holds then this N minus del g over del y will be the function of y alone there will be no x in this function N minus del g over del y.

So, with this now we can construct now this f and the f is nothing but the g here which we have constructed as the integral of this M. So, this g plus this function here N minus del g over del y which is a function of y alone and we are integrating here with respect to y. So, what we will prove now that this f actually solves the purpose meaning the differential of this f is nothing but the Mdx plus Ndy. So, this we will show now in next slide, ok. So, here. So, we will show that this del f is Mdx plus Ndy.

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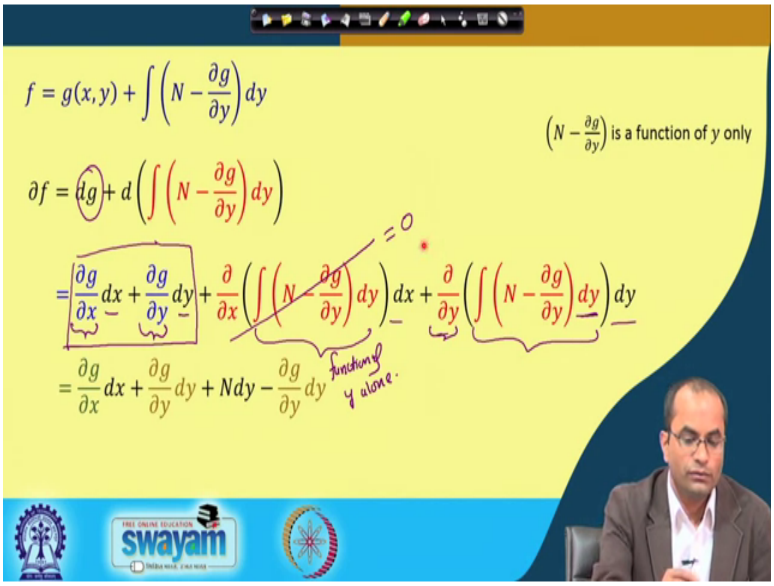

So, this is our f and if we take the differential of this f, so del f, so if we consider here this del f is del g plus the del of this function here which is a function of y. And now as per the definition this del g here, del g will be partial derivative of g with respect to x and partial derivative of g with respect to y; and we together with this dx and dy. So, that is the here dg the differential of g and now again the same definition for the differential of this one now we have this pico function.

So, del over del x of this function into dx and del over del y of this function and into this dy. So, now, this what we have observed that this is a function of function of y alone that we have already checked before that this is a function of y alone and its integral will be naturally again function of y alone. So, here when we take when talk about the partial derivatives with respect to x. So, this term will be set to 0 and here then we have del over del y and this is the integral here with respect to y and then we are taking the derivative term. So, this will remain only with this N del g over del y.

So, this is a function of y alone therefore, this goes to 0, and then we have this del g over del x dx del v over del y dy. And from this one because this was the derivative of this integral, so this integral and the differential will be removed and we will have N minus del g over del y multiplied by this dy. So, here we have N dy that will be one term, so N dy. The another term will be del g over del y into dy. And now again, so we have these two terms here they cancel out each other and then we get this del g over del x dx plus Ndy.

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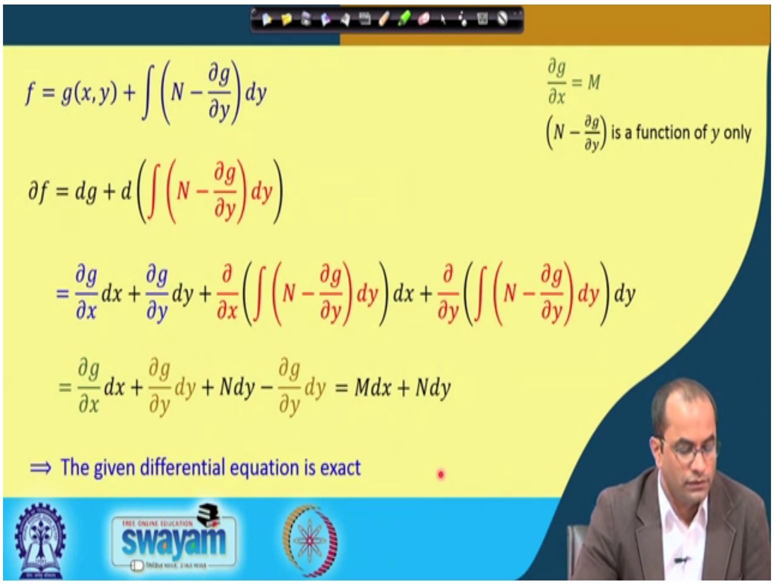

And this del g over del x is nothing but the M here, del g over del x is nothing but M. So, this here is M. So, what we get? Mdx and here this plus Ndy. So, Mdx plus Ndy this is what we want to show. So, this is the f here which solves the purpose here the differential of this f. So, there exist a f once we assume that this condition holds that necessary in sufficient condition that holds. So, that we can construct such a f whose differential is exactly equal to this Mdx plus Ndy.

So, we have seen now that this necessary and sufficient condition for the differential equation Mdx plus Ndy to be exact is that del M over del y must be equal to del N over del x. So, we need to check this condition for the exactness, and once we know that the equation is exact then we have to construct this f somehow whose differential is the given differential equation Mdx plus Ndy. And if it is not exact then we have to find some other way or to get the solution of the differential equation.

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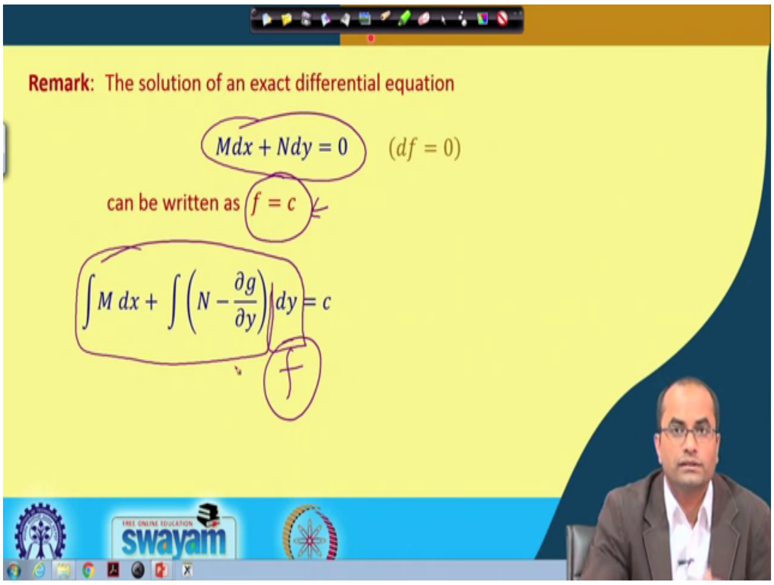

Well so, just a remark here the solution of the exact differential equation this Mdx plus Ndy. So, once we know that the equation is exact that means, this df is equal to 0 because there is a function f whose differential is this Mdx plus Ndy. So, meaning this df is equal to 0, and the solution then we can write down in terms of f that f is a constant because the differential of this function is 0. So, this function must be constant can be equal to the constant here. So, f is equal to see from this equation df is equal to 0 we are getting that f is equal to c. So, that is the relation we are looking for and this is the solution of this differential equation f is equal to c, this is the solution of this exact differential equation.

So, just to note again that this was the f as per our construction the integral of this Mdx was one term and then we have also this integral term this is f which we have just constructed before. So, this is our sorry here this is our f here, and we have constructed this theoretically. And what else we can actually get out of this?

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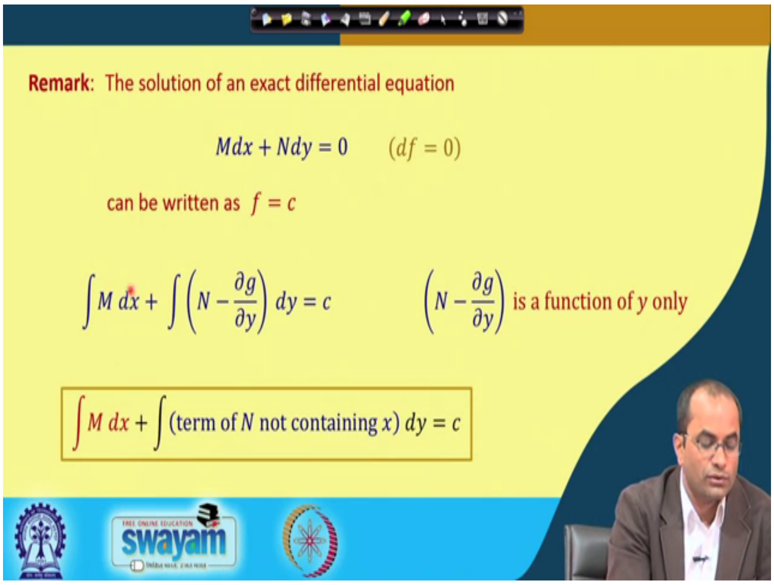

We have also notice or we have shown there that if this equation is exact then this N minus del g over del y is a function of y only. And then there is a directory also used in several textbooks. So, here we can write down directly for the given differential equation the solution once we know that the differential equation is exact. So, here the integral of Mdx, M is given. So, we can find out the integral of Mdx.

In this for the second part because this is a little bit complicated here to evaluate we have to find set up the g first and partial derivative and N minus. So, the directory case which comes exactly from here itself that we know that this is a function of y alone, so what we take from this N because N minus this something we have to remove it so that this becomes a function of y alone. So, the terms of N if we take here containing not containing x meaning that terms of this N which contains only the y. So, there should not be x here. So, only the terms of y if we take from N and then this integrate here that is exactly the solution of the given differential equation.

So, we will see in the example also that one can use this directory as well but there is no need because once we know that the equation is exact we have to just find a function whose differential is this and there was a another systematic approach which works to get this f here. So, we do not have to use this direct formula in that case.

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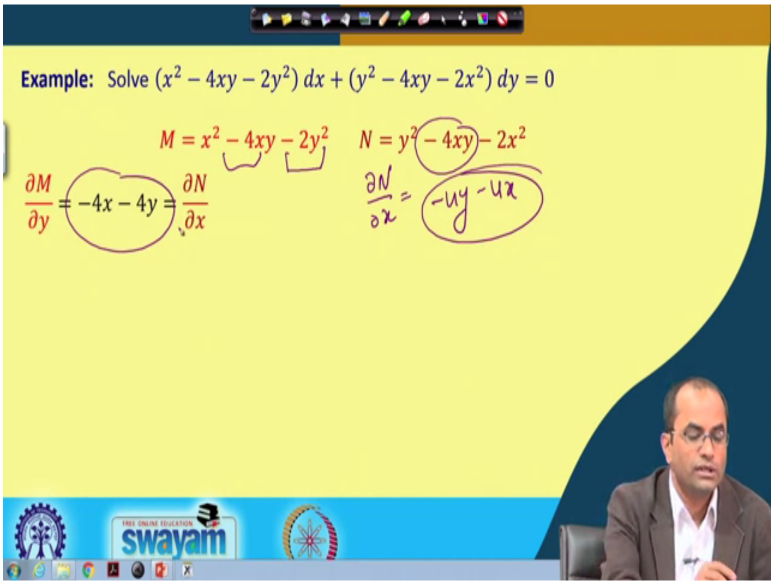

So, let us go through the example here we take this x square minus 4 xy e minus 2y square dx and plus this y square minus 4 xy minus 2x square dy is equal to 0. So, the M, the first part here is the M the x square minus 4 x y and minus 2y square and the second part here is N. So, we have this M and N in the given differential equation, first what we have to check whether the equation is exact or not. Meaning that we have to compute del M over del y. So, del M over del y will be minus 4x and minus 4y from here and this del N over del x will be minus 4y and minus 4x. So, both are same del M over del y and del N over del x and that is precisely the condition for exactness. So, the given equation is exact.

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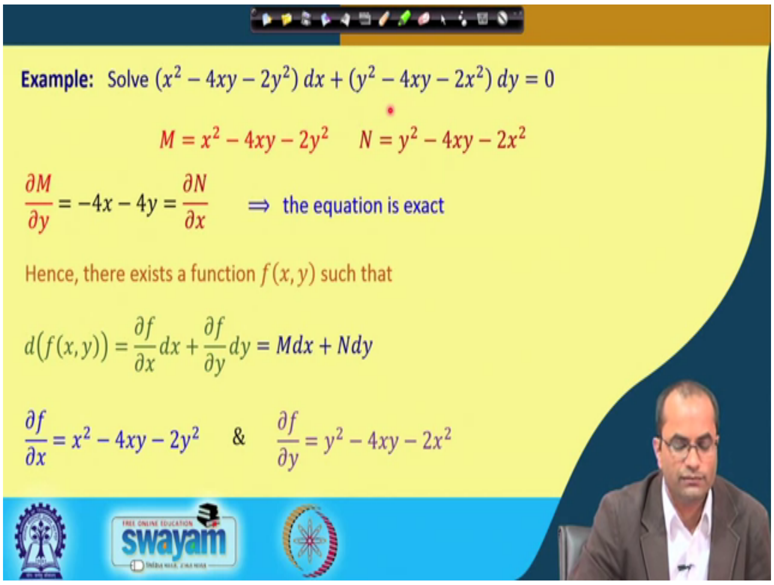

So, we know that there exist a function f whose differential is exactly given here this M dx plus Ndy. And now how to construct this f? So, that the trick is here that we know that df is equal to this one. So, we know the formula for the differential of this function f is its del f over del x, dx plus del f over del y that is the differential of this f.

And this must be equal to this given Mdx, Ndy. So, here we can have the relations of this f to this M and f to this M. So, we have here the del f over del x partial derivative of this f with respect to x must be equal to this M, meaning this x square minus 4xy minus 2y square and the partial derivative of this f with respect to y must be equal to the N which is y square minus 4 xy minus 2x square.

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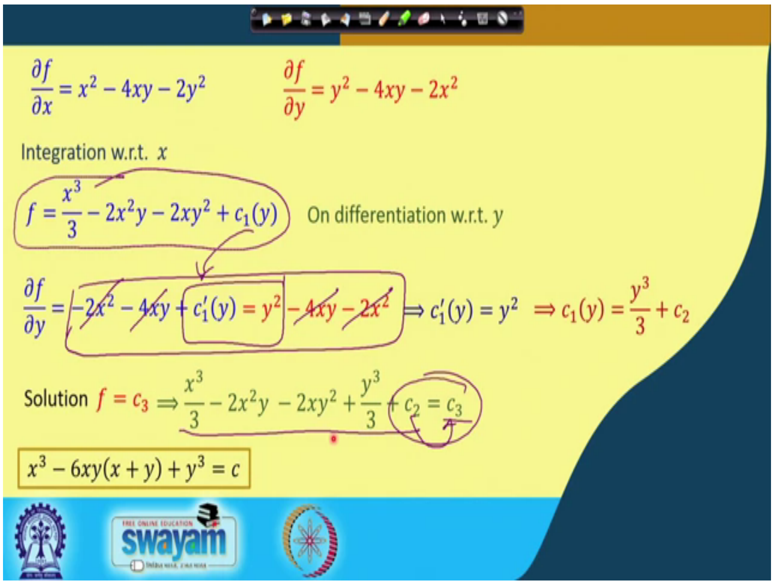

So, we have these two relations now with del f over del x given by this function del f over del y is given with this function. So, now, from these two equations we have to exactly find what is the f whose partial derivative with respect to x is given by this, partial derivative with respect to f y given by this. So, here if we integrate this with respect to x this first equation, what will happen? You will get this f as x cube by 3, from here we will get minus 4 y and this x square by 2. So, here minus 2x square y and here we will get minus 2y square and the x will come when we integrate it. So, this is the integral and that this is a constant of integration which in this case because this was the partial derivative with respect to x. So, that can be more general constant here could be a function of y itself.

So, we have integrated this del f over del x here to get this f, but there is the c 1 y is coming into the picture and we have not use here the another equation. So, now, from this f here from this given f which we have got by integrating with respect to x we will differentiate now with respect to x, so with respect to y and then we will equate to this given partial derivatives. So, on differentiation with respect to y what we will get here this term is x alone. So, this will be 0 here with respect to y, so we will get minus 2x square and here with respect to y, so we will get minus 4 xy and the c 1 y also here the derivative we will come.

So, this is equal to the y square minus this 4 xy minus 2x square that is given already here that the partial derivative with respect to y must be equal to this. So, out of this now here we can actually get this constant term easily this c 1 prime because that is a derivative when we have differentiated this here, so we got the derivative also. So, what we see here that minus 2x square lies both the sides here and minus 4 xy is also there. So, what we get? We get that c 2 prime y is equal to y square. So, we get this condition that the derivative of this c 2 with respect to y is y square. And now we can integrate this one, so we get c 1 y is equal to y cube by 3 and a constant here. So, meaning once we get the c 1 we get basically this f. So, f is nothing, but x cube by 3 minus 2x square y minus 2xy square and then we need to add the c 1 there, y cube by 3 and plus a constant and arbitrary constant they are now.

So, once we have f we can write down the solution as f is equal to constant. So, the solution will be f is equal to some constant c 3 we have named it because c 2 we have used already. So, c 3 and that means, now this is you our f by substituting the c 1 here in this equation in this f equation here. So, this is f here and with c 2 is equal to the c 3. Now, the c 2 and c 3 they are two different arbitrary constants, but we can merge into one and meaning that this is the solution of the equation.

Indeed, we have multiplied by the equation this by 3 here. So, to get x 3 minus 2x square minus 6 x square y and also this minus two xy square multiplied by 3 again. So, that will be also 6 xy square and plus this y square when multiplied by 3 and the c 3 minus c 2 we have taken another constant here with multiplication by this 3. So, that is the solution of this differential equation which we have considered. So, once the equation is exact the process exactly follows what we have taken here we need to equate this del f over del x is equal to M and del f over del y is equal to N and from there we just integrate this then differentiate it equate equal to the given derivative from there we will get this c 1 prime y which again on integration we will get c 1. By substituting the c 1 we got f and then we can write down the solution of the given differential equation.

Now, the question is if we follow the same process here when the equation is not exact then at what point we will get the problem or where we will stuck, and where we will realize that such f does not exists.

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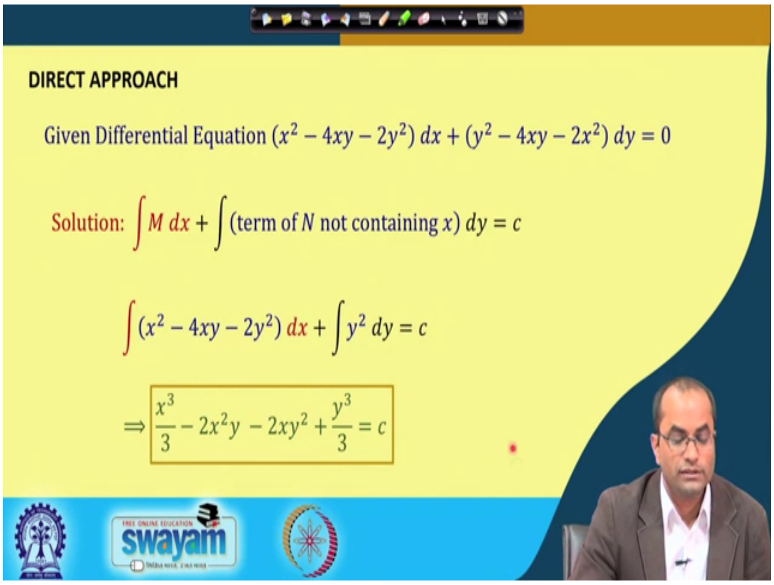

So, if by mistake, before we go to that point here using that direct approach which we have written down before that we have a direct formula as well. So, this is the given differential equation and the solution we can also write down directly in this way that integral Mdx and this terms of N not containing x and dy. So, here this M given in the differential equation and then the terms of this N not containing the x that is only the y square term. So, we have y square dy is equal to this c and on this integration we got exactly the same solution which we have obtained before. So, this was one can use this formula to get the solution of exact differential equation.

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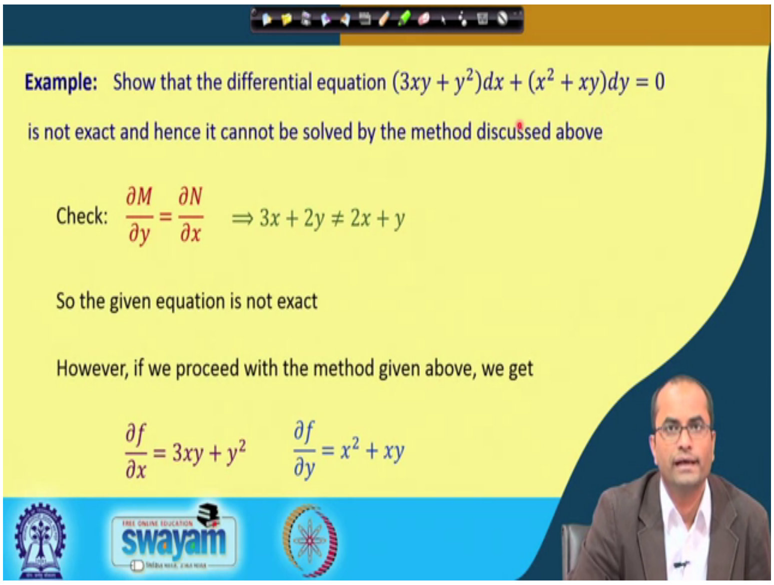

And now what we will now get back to the discussion which I have initiated just before that if the question is not exact. So, for instance we will consider this equation and we will first see that this equation is not exact and hence it cannot be solved as the method discussed above. And the main point is we will see that where we will get the problem now if we assume that this is the differential of some function f, then exactly at what point we will realize that the equation is not exact, if for instance we did not check the condition here for the exactness and just proceed to get air force differential is given differential equation then what will happen.

So, just to check here del M over del y is equal to del N over del x this is not the case. So, they are not equal. And then so the given equation is not exact differential equation, but you will now proceed exactly how we have done before that means, this del f over del x we will take 3xy plus y square and this del f over del y as x square plus xy.

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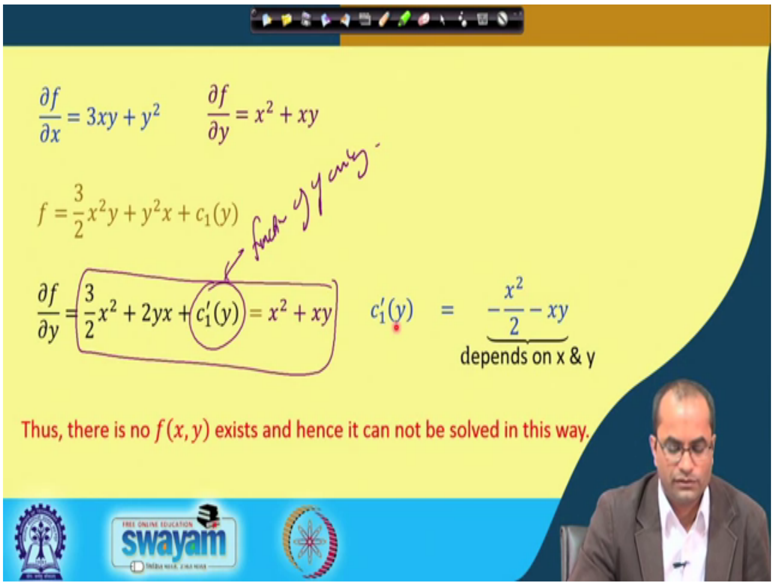

So, with these two conditions we will now integrate this one with respect to this x here. So, f is equal to 3 and this x square by 2y and this xy square and c 1 y. So, just on integration of this we got this function and then we can differentiate this with respect to y and can equate to this one. So, once we differentiate here with respect to y, so we will get 3 by 2 x square here 2 y x and then c 1 derivative y, and now we equate to this x square plus xy.

Now, on equating this one last time if you if you remember we got the c 1 prime y as a function of y and then we integrated and we got the differential equation. So, this is c 1 prime y, the partial the other derivative of the c 1 with respect to y the ordinary derivative of c 1 with respect to y there is only this is a function of y only. So, the function of y only. But when we equate here what are we getting now? c 1 y is equal to this minus x square minus xy, so there is a contradiction. On the left-hand side we are telling that this is a function of y the right-hand side we are also getting x in this equation, while in the previous one when the equation is exact we will never get this as a function of x here we will get only y terms.

So, here this depends on x and y, and we cannot solve this equation because this is inconsistent now. Here we are talking about this is left-hand side is telling that this is everything function of y and the right-hand side we are we are having the function of x and y. So, naturally we cannot solve because this is inconsistent equation and hence we cannot find such a f whose differential is the given differential equation. So, this was just to demonstrate that if by mistake without checking the exactness we proceed here we try to find such a f whose differential is this given differential equation then we will stuck exactly at this point where we cannot proceed further and hence such f does not exist.

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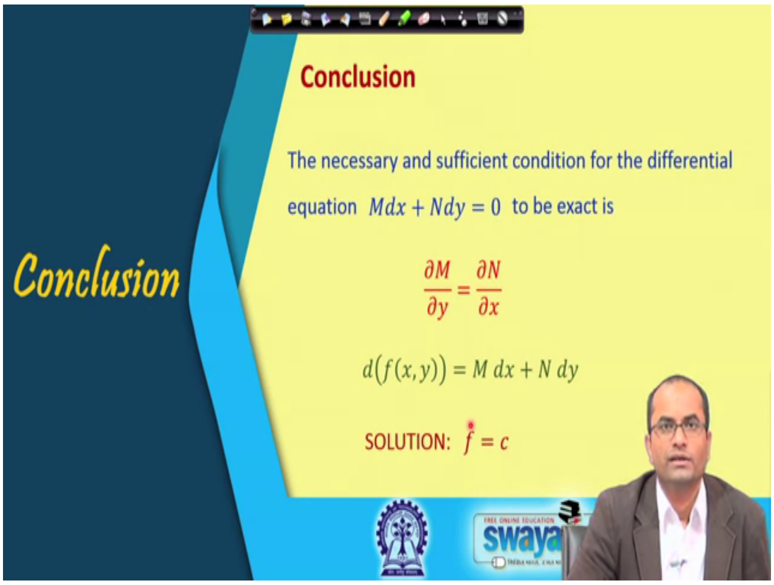

So, the conclusion is that the necessary and the sufficient conditions for the differential equation to be exact is del M over del y is equal to del N over del x and once we know that the equation it is exact then we can find a f whose differential is this Mdx plus Ndy and then we can write down the solution as f is equal to some constant. So, that is the process for finding the solution checking the exactness of the differential equation. And in the next lecture we will continue if the given differential equation it not exact then what method we should process to get this solution.

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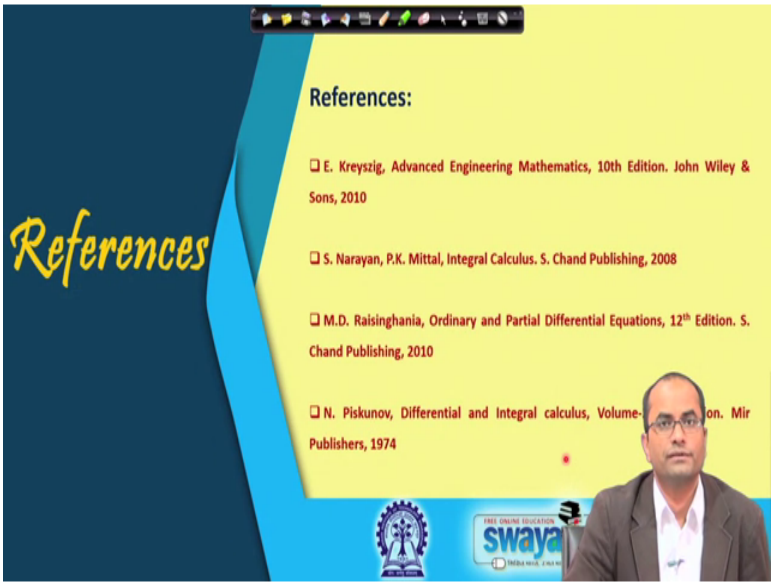

These are the references used here, and.

Thank you for your attention.