Engineering Mathematics - I
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Lecture - 28
Double Integrals

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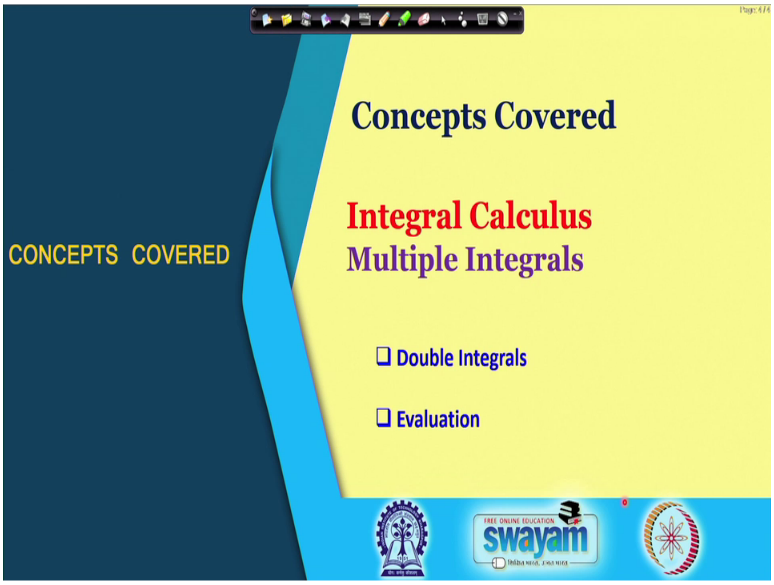

So, welcome back, this is lecture number-28. And today, we will be talking about Double Integrals. In particular, we will go through some simple cases of evaluation and before that we will introduce the double integrals and their its properties.

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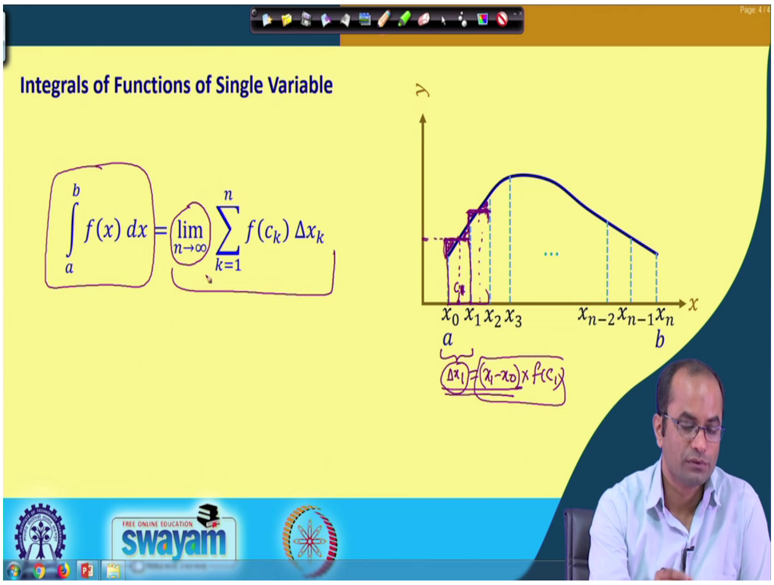

So, the integral of functions of single variable, so before we define for the two variables. Let us just recall the definition, what we had for the single integrals. So, integral a to b this f x dx was defined as the limit of this sum. So, what was this sum, if we just take this function f x which is defined from a to b, this is x axis and we have your y axis.

So, if we divide this domain from a to b into several pieces, so let us denote here the first point a by x 0, then we have a x 1 here, x 2, x 3 and so on, x n minus 2, x n minus 1 and x n. So, we have divided the whole range a to b into n intervals. And then in each interval we have taken a point, it could be a middle point or it can be any other point here. So, this c k point and then so c k is between x 0 and x 1 in this interval.

And we take the value at this point of the function, which is here. And so f c k into multiply by this delta x k. So, this is delta x 1 in our notation. And this is c 1, because this is the first interval and this will be the f x f c 1. And we are now integrating this summing this product. So, what is this product, this product will give the area of this rectangle here.

So, this width x 1 minus x 0, which is delta x 1 and multiplied by the height, which is f c 1. So, and then we are summing all these cases, so all these rectangles. So, we will get finally the area of these rectangles in the finite sum. But, when we are taking the limits, so in the limiting case when n is approaching to infinity. So, this error for in each rectangle there is a error part here, which we are for example here we are taking more area then the area under the curve perhaps or the less we do not know. Here also the same situation, so they could be in error, because we are taking the area of these rectangles.

So, in the limiting case, when these intervals the length of these intervals are going to 0. So, in that case this limit here will approach to the area under the curve. So, this integral a to b f x dx represents the area under this curve here and this is defined as the limit of this sum. So, similar concept we have for the integral of two variables or the double integrals.

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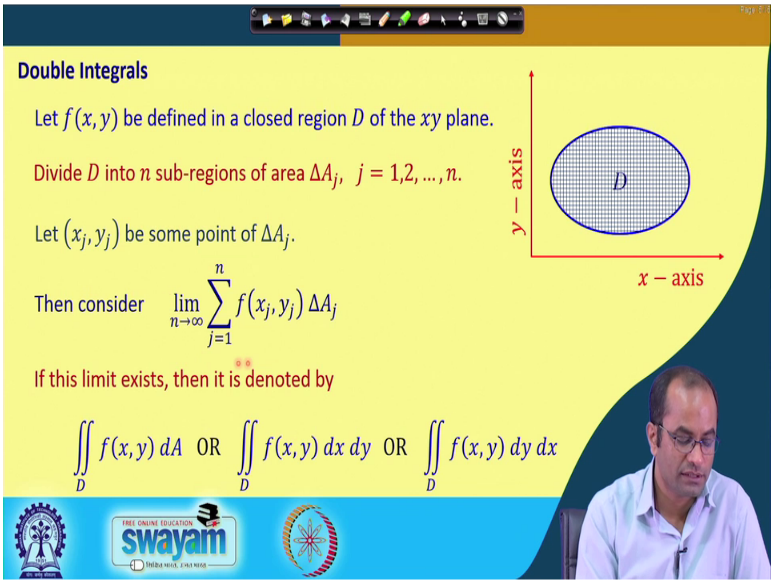

So, let us just go through. So, in this case let f x be defined in a closed region d of the x, y plane. So, this is the region here we have define a closed region D, it can be of any shape in general. So, here we have x axis and the y axis and then we divide again this region into sub regions so into a smaller regions of area delta A j. So, the each the sub region will have area delta A j and j varies from 1, 2, n.

So, such rectangles or the squares are actually n and each of them has the area delta A j. So, if we take a point x j, y j are some point in the area delta A j a general point, we are calling this a general square or the rectangle, we are calling as delta A j and we take a point there at which is denoted by this x j, y j.

And then we consider this sum, so we are summing here again the value of the function at that point. So, here we have taken some point x j, y j and then there will be a function defined as a surface in the third dimension, which is usually taken as z axis. So, there will be a surface. So, on the surface you will have this point there f x j, y j corresponding to this point x j, y j and we have multiplied by this area, this delta A j.

And if this limit exist, then we denote this integral this value of this limit by this integral over D, so the double integral D. And f x, y and this d A or sometimes you also used the notation sheet of d A, which is representing this area of this infinite symbol sub region. So, we can also denote by dx, dy or dy, dx, so that is the definition here again its the limit of this sum, which where we are adding actually this f x j, y j and multiplied by the area of the smaller region in the x, y plane.

And the there is a point x j, y j in each sub region in corresponding to this point, we have the function value f x j, y j. Adding all these and then taking the limit as n approaches to the number of these rectangles goes to infinity. And in that case if this limit exist, we define this by such integral notation.

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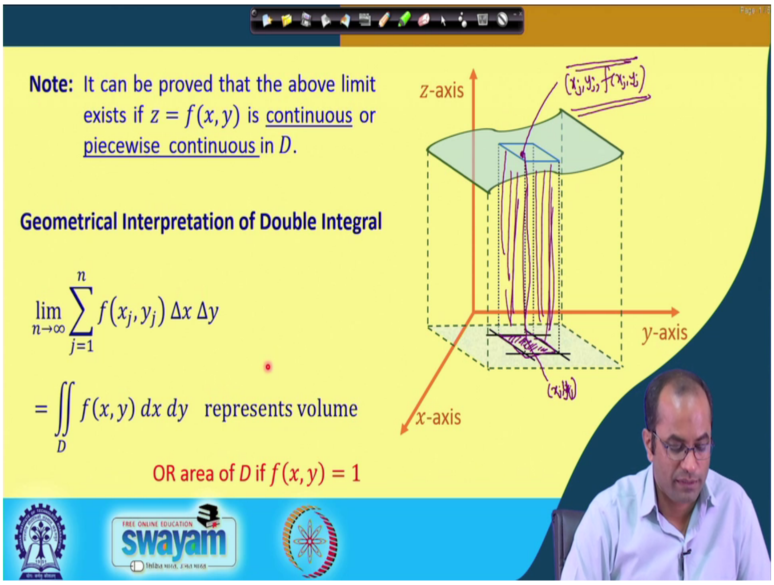

So, it can easily be proved that this above limit exist, so or this integral exist. If this function is continuous or piecewise continuous in D, so we have this closed boundary domain and if the function is piecewise continuous or continuous, so this is sufficient for the existence of such integrals.

And coming to the geometrical interpretation for these double integrals. So, this is also clear from the limit definition, which we have seen. So, we have taken the small region in the x, y plane or the whole domain in the x, y plane was divided into many sub regions. And this consider for example, one sub-region here and these corresponding to this we have the surface here, which is actually the projection given by this one.

So, the double integral. So, we will take a point in this sub region at some point we will take which we can denote by usually x j, y j. And then corresponding to this point, we will have a point there in the on the surface here. So, we have this point like x j and y j, and then f x j, y j.

And there we take this product of the area of this rectangle or the sub-region in the x, y plane and multiplied by this height. So, what do we get, when we have this area here and then we have multiplied by some height. So, we will get again this volume of this sub-region, which is define here. And then we add such sub regions and take the limits. So, we will get just the area under this the volume of under the surface in this case.

So, this limit here, because this delta x delta y for example this was the area of this triangle or the sub region defined here, which is multiplied by this f. So, this gives the volume of this smaller sub region and such sub regions we are adding here and that will be denoted by this integral, so that will represent the volume.

And in case for example we said this f x y is equal to 1, so what we will get, so if we set here f x j, y j is equal to 1, so we take the function 1. So, the same value will also give the area, because we are basically adding delta x delta y. So, we are adding all these areas, and we will get the total area of the domain D. So, in that case the area of D if we for example take this f to be 1, so why with this double integral, we can get the area of the domain in the x, y plane or basically this represents the volume under that surface over the x, y plane well.

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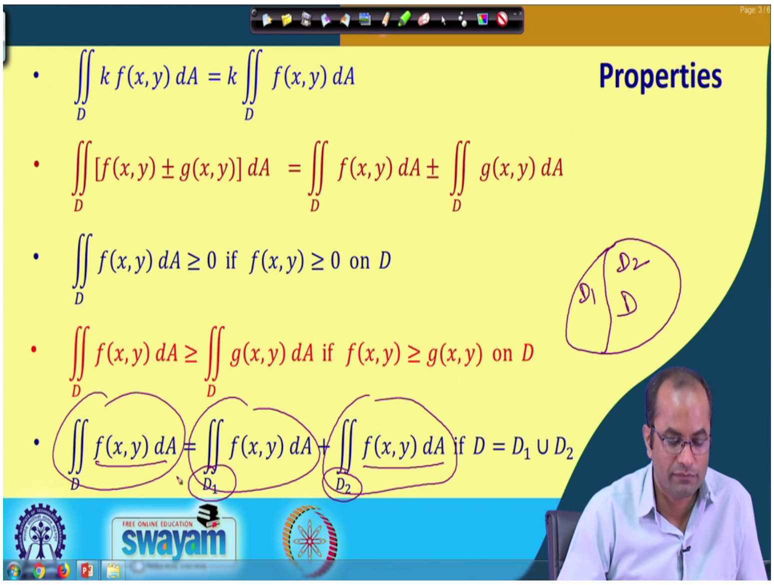

So, coming further now for the evaluation part, before we go to the evaluation let us discuss some properties of the double integrals. So, similar to the single integral, we have more or less the same properties for the double integral as well. So, for example we have here the k multiplied by this function f x, y and in that case this is a constant k, so that can just come out the integral, and then we have this integral d f x, y d A.

Similarly, we have the addition here or the difference of the two functions f and g. So, in that case this integral will be over this f and plus or minus the integral over D. So, another property of this double integral. So, here if we take the function as a non-negative on the domain D, and we are integrating here this over D, then this value of this integral will be also greater than or equal to 0.

Similarly, if we have two functions f x, y, which is greater than the function g x, y or its this f is taking move the larger values then the g on the domain D. In that case, the integral of this g, f will be also greater than the integral of this g here. So, the last one which we are discussing here, so that is the additive property. So, we have this integral over D and this domain if we break into two parts.

So, we had for example the some domain here, which was D and we have defined I mean divided this into the domain D 1 and then D 2. So, in that case this integral over D, we can write the integral over this D 1 and then the integral over D 2 the same integrant, same integral only this region has changed. So, this is over D 1 and this is over D 2, so that is also common property of the of this single integral. So, we have more or less the similar properties for the double integral.

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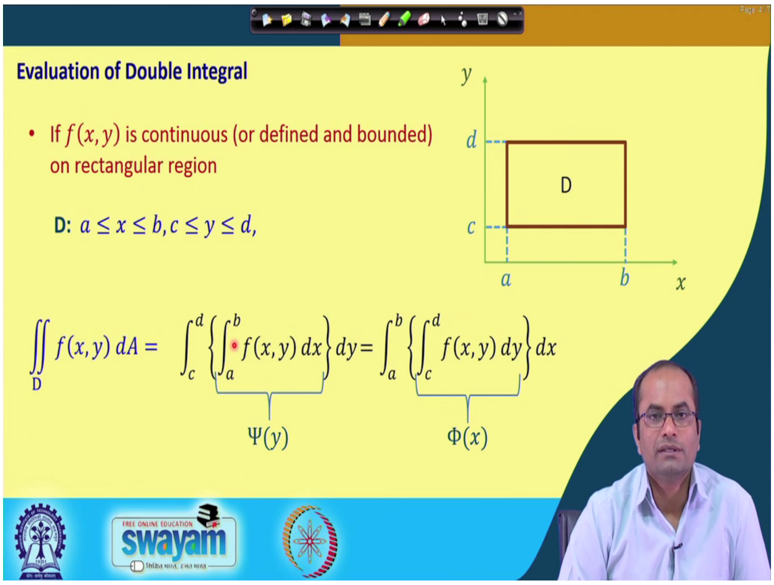

Now, coming to the evaluation. So, for the evaluation, we have we will consider first the a simple scenario. So, for example, if this f x, y is continuous or defined and bounded in that case also this integral will exist on this rectangular region. So, we have taken we have restricted our self to the rectangular region. So, x varies from a to b and y varies from c to d.

So, for this rectangular region, which is also given in this figure. So, this rectangular region is from a to b for x and in the in the y axis, it varies from c to d. So, we have these constant numbers a to b, and there also c to d. So, this is the simplest case, where we can define this integral over the such a nice domain, because the limits are clear that for x, we are wearing from a to b. And in the direction of y we are wearing from c to d and the limits are constant they are not depending on each other.

So, here this integral over this. So, over such a rectangular region d A will be defined as so first we will compute the integral with respect to x. For example, we could do with respect to y first does not matter in this case. So, let us just discuss this one first. So, we take this integral f x, y dx over this dx treating this y as a constant. So, taking this integral here from a to b, because x varies from a to b. So, we have taken this integral a to b. And now this will become a function of y, so because we have integrated over this x from a to b. So, the y will remain as it is so this will be a function of y and then we can so this is a repeated integral.

So, now we can integrate with respect to y, so y will vary from c to d. And since the integral limits here, because the region was nicely define from a to b and the c to d in the direction of y. We can also compute first the integral with respect to y so this dy. And then y varies from c to d or which we can call like phi of x, so because this will be a function of x after the integration. And then this is a function of one variable and we can now integrate over the x, so the x will vary from a to b.

So, in this case it does not matter whether we first compute with respect to x and then with respect to y or first with respect to y and then with respect to x for such domain. And here the getting the limits are also much easier, but this is not always the case, because we do not have a such nice domain all the time.

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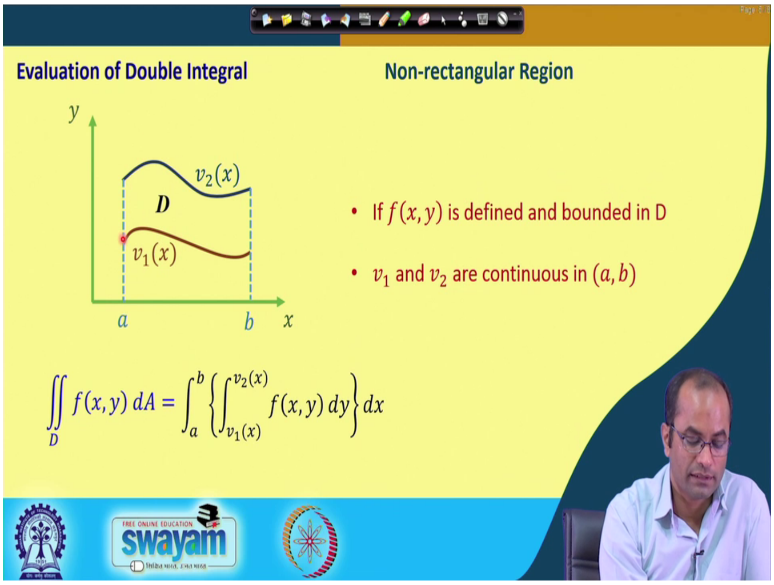

So, for non-rectangular regions. Again we will consider a some simple cases and then later on we will come to the more general cases. So, here for example have a domain is of this kind. So, this is the domain here. So, our domain of the function so, our function is defined this domain. So, correspondingly we have the function values there.

So, for this domain here and this domain is bounded by this v 1 x. So, here we have now function v 1, which depends on x. So, as the x varies, we have the curve here. And the same thing, we have some other v 2 x function with respect to x, so that is boundary here also changes. In the direction of y, we have this constant. So, at this from x is equal to a to this x is equal to b, this is the cut. But, in this direction here in the direction of y we have the functions v 1 x and their v 2 x.

So, for such domains also it is a not difficult to compute the integral over such over this domain D. And first we assume that this f x naturally is as defined and bounded or discontinuous or piecewise continuous in this domain. And this v 1 and v 2, they are the continuous functions. So, we have this domain, which is depicted in this figure.

And then this integral the double integral of this f x over this domain D, we will take now so we have to now follow this sequencing. So, first we have to integrate with respect to y, because now with respect to y we have so for getting this limit this is the easiest way to understand. So, of with respect to y draw a line this parallel to y and then we see that this line here enters at this v 1 x and go out at v 2 x. So, these are the limits now in the direction of y and then such lines they goes from here x is equal to a to and everywhere up to x is equal to b.

So, this is the first integral, which is taken over y. So, for y direction we have the limits here v 1 x v 1 x to this v 2 x. So, v 1 x to v 2 x in the direction of y. So, having this integral first this again a single integral, because this x we are not touching here, we are integrating with respect to y. And then taking these limits also from y 1 x 2 v 2 sorry v 1 x to v 2 x.

And then once we are done with this integral, we will integrate with respect to x. So, for x now the limits are from a to b. So, for the x limits are constant here a to b, but for the inter inner integral y the limits were depending on x. So, v 1 x to v 2 x. So, for such domain naturally we will follow the sequencing because if we go the other way round that first we integrate with respect to x, then there will be a problem, because we do not have a such a nice representation of this whole integral so or whole domain here. So, in this case the y we have the nice limits here v 1 x to v 2 x. And for then x we have the constant limits from a to b, so that is the sequence, we should follow for such domain.

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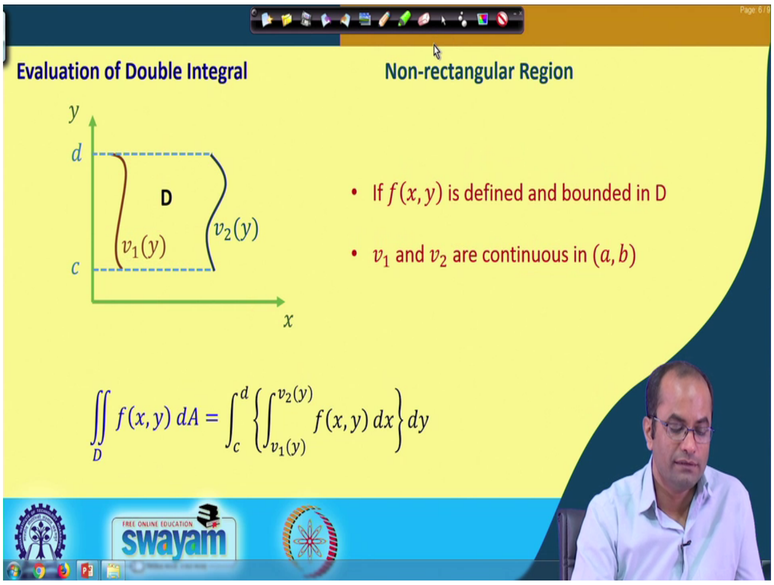

And if we have for example the domain given in this figure, so here there is a curve which depends on this y. So, we have v 1 y to this v 2 y and then in the y we are going here from c to d. So, this is just the other way round, then the earlier one. And for this case also, so we have this f 1 access why is defined and bounded and v 1 and v 2 are continuous.

So, in this case this integral of this f x, y over such domain will be now we will integrate first with respect to x. So, in this case we will now discuss, now we will take first with respect to x. So, here therefore the limit we are going from this v 1 y to v 2 y. And such lines now will vary from this c, they will go from c to d, so that is the way we can compute the easily put the limits for x and y.

And in this case, so here the first integral as I said, we will compute with respect to x now. So, for x we have v 1 y to v 2 y, we have v 1 y and to v 2 y. And once we have this integral, which is will be a function of y. So, we will integrate with respect to y then and y the limits will be c to d. So, again in this case we have to see which one is convenient now.

So, we have this first we need to get the integral with respect to x, because getting this limits are as much easier now and for the while we will put the limits from c to d. So, in this case also we have seen that we can easily evaluate the integrals, because once we have these limits we are going back to the single integrals, which we know how to evaluate.

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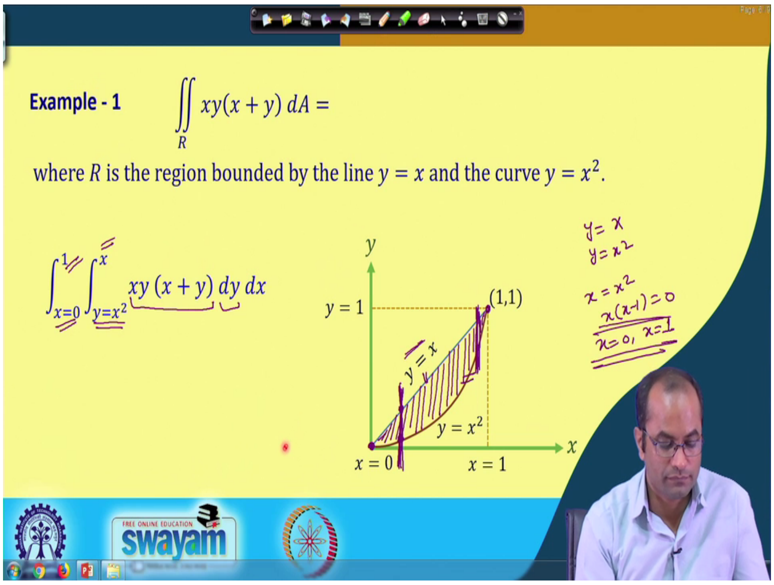

So, let us go to the example here. So, the first example we have this integrand x y and multiplied by x plus y d A. And we will compute what is the value here. And this region is R is bounded by the line y is equal to x and the curve y is equal to x square. So, we have this line here and the parabola y is equal to x square.

So, if we draw I mean this is the first step that we have to draw the region of integration, and sometimes that is the difficult part. So, here we have y is equal to x line, this y is equal to x line and then we have y is equal to x square this parabola. So, they intersect because y is equal to x and y is equal to x square, if we solve here. So, y is equal to x we put, so y is equal to x. So, here we have x and x minus 1 is equal to 0. So, x is equal to 0, and x is equal to 1 at these two points they intersect.

So, at x is equal to 0 we have the y also 0 naturally, so this is the point of intersection. And the other one when x is equal to 1, so y is also 1 so, this is the another point. So, this is the area which we call the area of integration. So, over this region or the region of integration, we will now integrate the given function this x y multiplied by x plus y and this d A. So, having this now we have the possibilities that we first take the integral with respect to y. So, while taking the integral with respect to y, we will take so with respect to y.

So, we will draw a line here parallel to the y, y is equal to y axis, so this is the line. So, now you will see that this enters here at x square and it leaves the domain at this y is equal to x line. And that that is true everywhere in the domain at whatever point you draw this line parallel to the y axis, it will enter through the y is equal to x square, and it will leave the domain at y is equal to x line. So, and then such lines will vary from x is equal to 0 to x is equal to 1.

So, first we will put the limit here for y, because we are in going to integrate first with respect to y for x for instance in this case now. And so here the limit for y will be x square, because at this point when it enters the domain it is x square. So, here we have this y is equal to x square. And then when we leave the domain, this is y is equal to x, so we have y is equal to x. So, these are the limits we have the integrant here x y x plus y, and we will integrate with respect to dy.

So, once we integrate this one, then we have to integrate then with respect to x. And the x limits are clear, because these lines which we have drawn here. They are going from x is equal to 0 to x is equal to 1 and then we are swapping the whole domain. So, here from x is equal to 0 to x is equal to 1, these are the limits for x. So, x is equal to 0 to x is equal to 1 or what we can do we can take the limits other way round as well.

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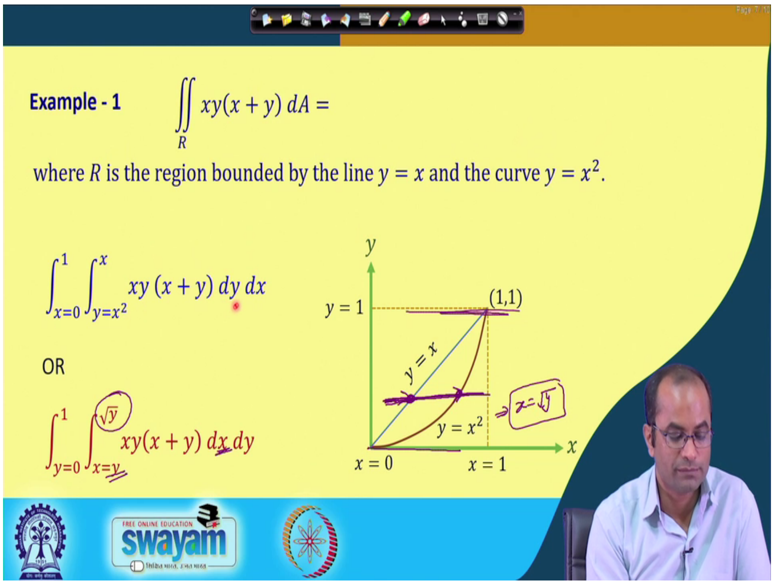

So, if we take if we consider for example first with respect to x, so what we have to do? Let us draw a line parallel to this x axis. And this line will is entering here at y is equal to x and living this domain this region at y is equal to x square this parabola. So, while entering here, so we are putting the limit for x. So, here the x is equal to y that is the entry point, so x is equal to y that is the limit. And it is leaving at, so here from this y is equal to x square, we can take y is x is equal to square root y.

So, we are leaving the domain here at x is equal to square root y, so these are the limits now for x. And now such lines will vary from this y is equal to 0 to y is equal to 1, so that will be the limit for y. So, for y limits here we have y 0 to 1 or we have for this x limit we have y to square root y. So, in this cases we have either the possibility of taking first with respect to y, then with respect to x or the other way round.

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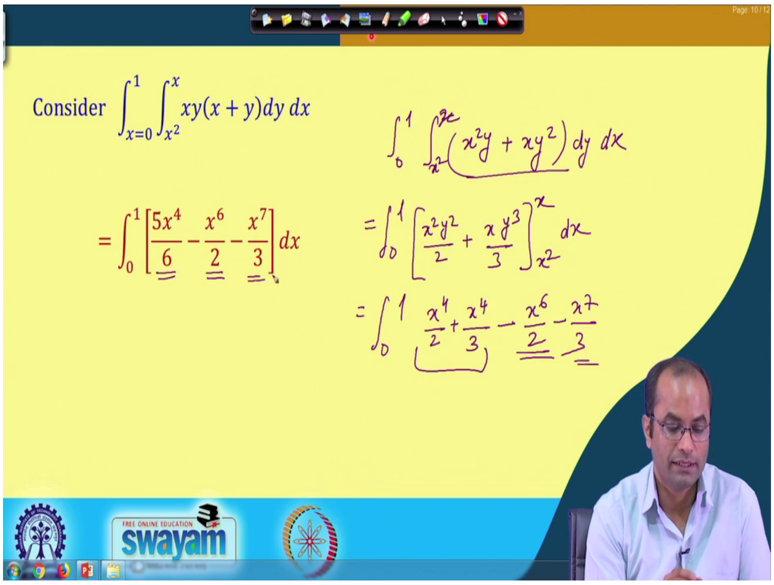

So, let us take the first one and then compute this integral. So, we will take the first one. And then taking this one we will integrate now. So, after this integration of the inner integral, we will get this. So, this is how we are doing. So, this integral of the inner one, so we have this integral here 0 to 1 and then x square to x the integrant is x square y and plus we have x y square with respect to dy and then dx.

So, this is 0 to 1 and this we will integrate now with respect to y. So, we will have x square and y square by 2, then we will have x and y cube by 3 that is the integral and then we will put the limit a from x square. So, x square to x and then we have the outer integral dx. So, this will be 0 to 1 the one we putting x there. So, we will have x power 4 by 2 and then we will have here also x power 4, because y is substituted this x is substituted for y. Then 3 and then minus for this x, so we have here x when we substitute so x square sorry. So, you will be substitute x square, so you will have x power 6 and by 2 and we will also get here then x power 7 x power 7 by 3. So, now this one it is 5 x power 4 5 x power 4 by 6 that is a term here x power 6 by 2, we have here and x power 7 by 3, we have the term there.

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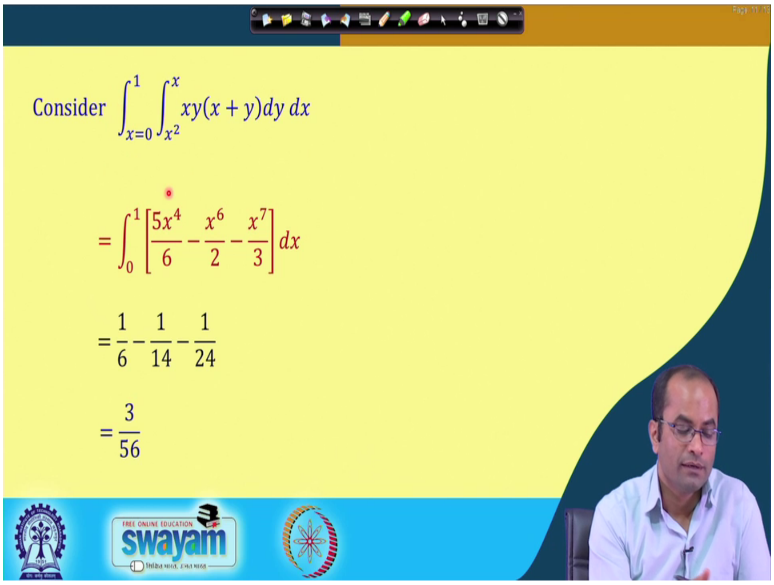

So, this is after the first integration. And now again this is a single integral, so with respect to x, so we can integrate again this. And we will get these number, because from here we will get x 5 by 5 and this 5 will get cancel, we will get 1 by 6. Here we will get x power 7 by 7. So, when we put the limit here only this x is equal to 1 will contribute, because at x equal to 0 will make this 0. So, here x 7 by 7 so, 7 into 2 it is a 14 and then that x 7 when we put the limit that will become 1.

Here also we have x power 8 over 8, so this is 8 into 3 24 and that is the one there. So, this is a value we can simplify this. So, it will be 3 by 56, so that is the value of this double integral by taking the integral first with respect to y and then with respect to x what we can do the other way round as well.

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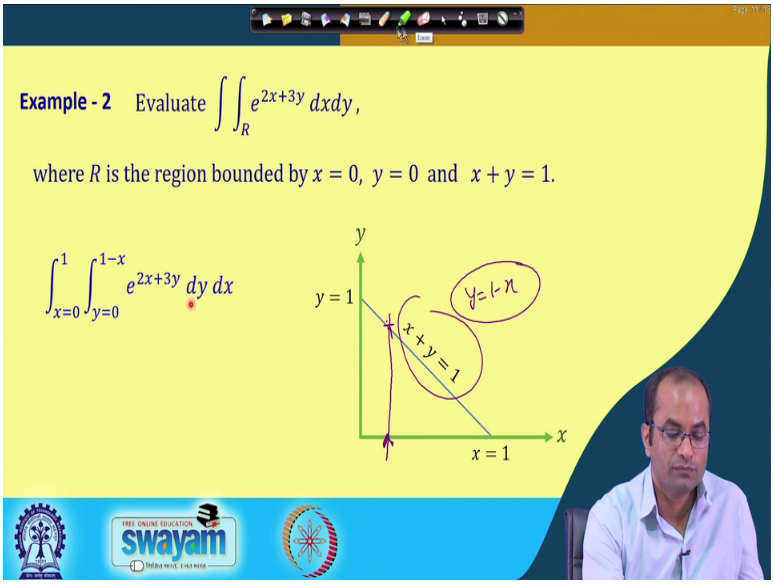

So, one more example here, we want to evaluate now this e power 2 x plus 3 y over this region R. And R is the region bounded by the x axis, so x is equal to 0 that means, the y axis y is equal to 0 we have the x axis and then there is a line x plus y is equal to 1. So, the first step always it should be the sketching the domain. So, if you put the domains, so we have x axis, we have y axis and then we have the line here x plus y is equal to 1. So, this is a triangular domain and again very simple to evaluate the limits whether we take first in the direction of x or we take first in the direction of y.

So, suppose we are taking the direction of y first. So, now draw the line parallel to the y axis. And then we will see that this line enters here at y is equal to 0 and this leaves the domain at y is equal to so from here we will have y is equal to 1 minus x. So, the limits will be y is equal to 0 to y is equal to 1 minus x. And then the x limits will be from 0 to 1, so then x from 0 to 1 and then the dx, so that is the one way of computing the integral. But, in this domain again it is a same similar limits we can get, when we integrate first with respect to x.

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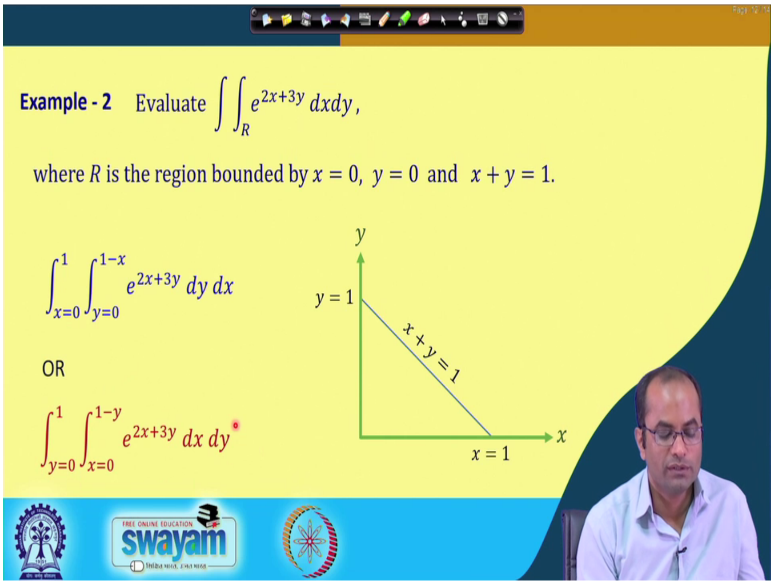

So, for integrating first with respect to x, we will have the limits for the x, so we will draw a line parallel to this x axis and that enters here x is equal to 0. And leaves here at this line, where the value of x will be now 1 1 minus y. So, x from here will be 1 minus y. And then we can integrate with respect to y, so the y will go from 0 to 1, so y will be from 0 to 1. And then either we can take the first integral or the second one to compute.

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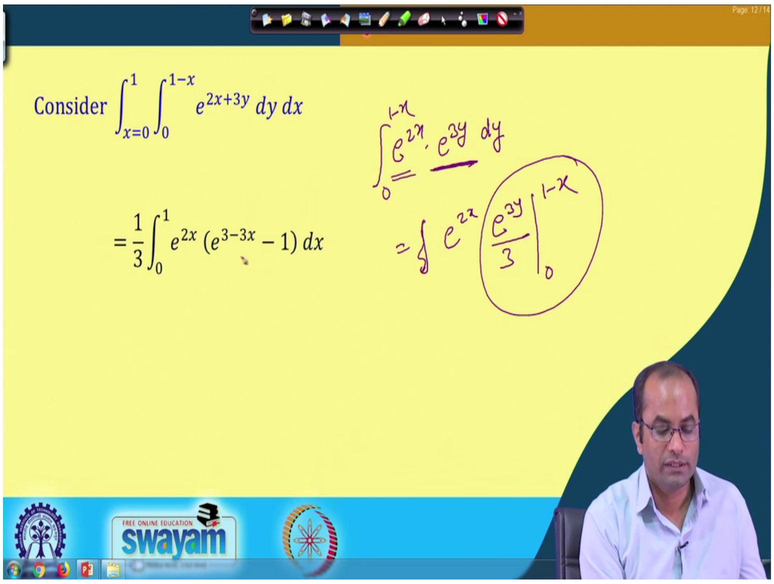

So, let us take the first one. So, taking this one here 0 to 1 minus x, and this x goes from 0 to 1. So, we will integrate this the inner one with respect to y. So, we have here e power 2 x and then e power 3 y that is our integrant. So, we are integrating with respect to y that means, after this integral of this 1 minus x dy the inner one. So, we will get here e power 2 x is a as a constant with respect to this y and e power 3 y you will become e power 3 by over 3 and then this limit 0 to 1 minus x. So, when we put the upper limit here, we will get e power 3 minus 3 x and then minus 1 here and e power 2 x is sitting here.

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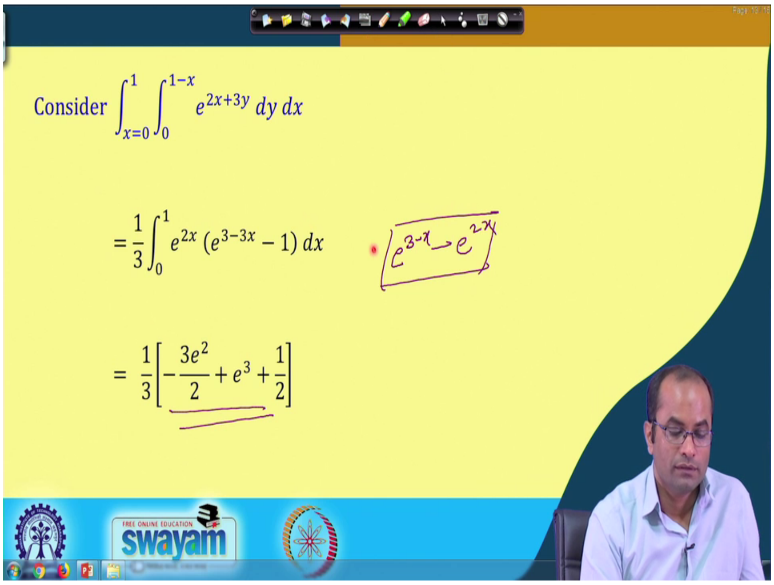

So, this is the first integral the inner one. And now we can integrate this as well because the single simple integral. So, we will get this here 1 by 3 and then they will be minus 3 square by 2 plus e cube and plus half. When we integrate this simple integral, you can multiply here by this e power 2 x. So, we will get e power 3 minus x and then minus here e power 2 x, which we can integrate and put the limit 0 and 1 and we will get this number here.

So, this when we have the simple domain like the rectangular triangular or when we have seen with which was bounded by the line and the parabola. So, in these cases finding the limits whether first with respect to x and then with respect to y or with respect to y first, and then with respect to x, in either way it is simple to get the limits. And we can evaluate the integrals by this repeated a single integrals.

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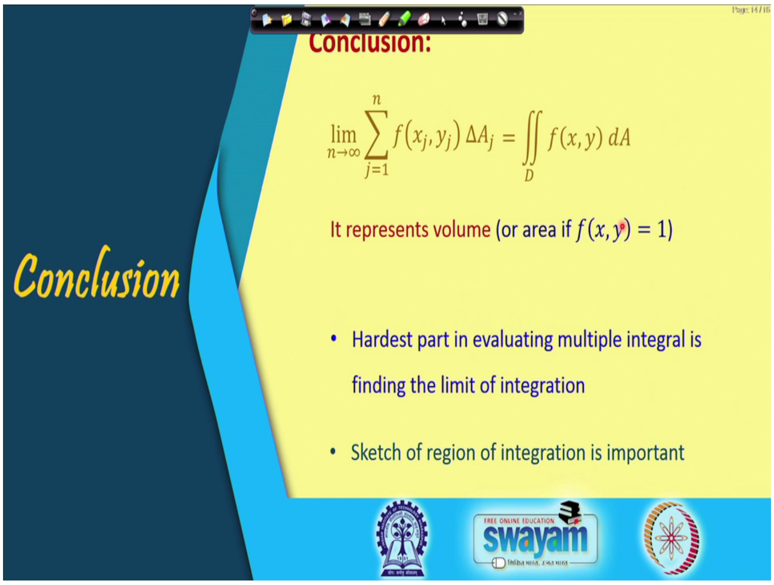

So, the conclusion is we have seen the the definition also for the double integral and which represents the volume or if we said this function to be 1. It can also represent the area of the domain d. The hardest past hardest passed usually is the evaluating multiple integral is the finding the limits of integration, but in cases which we have considered today there was simple a geometry and we can easily find the limits of integration. And the sketch of region of this integration is important, because without sketching the region we cannot get the idea of the limits of the domain.

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So, these are the references which were used to prepare these lectures.

And thank you very much.