Engineering Mathematics - I
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Lecture – 24
Improper Integrals (Contd.)

So, welcome to the lectures on Engineering Mathematics-1, and this is lecture number-24. And we will be talking about this improper integrals. So, in particular we will discuss today the convergence of improper integrals of type-2.

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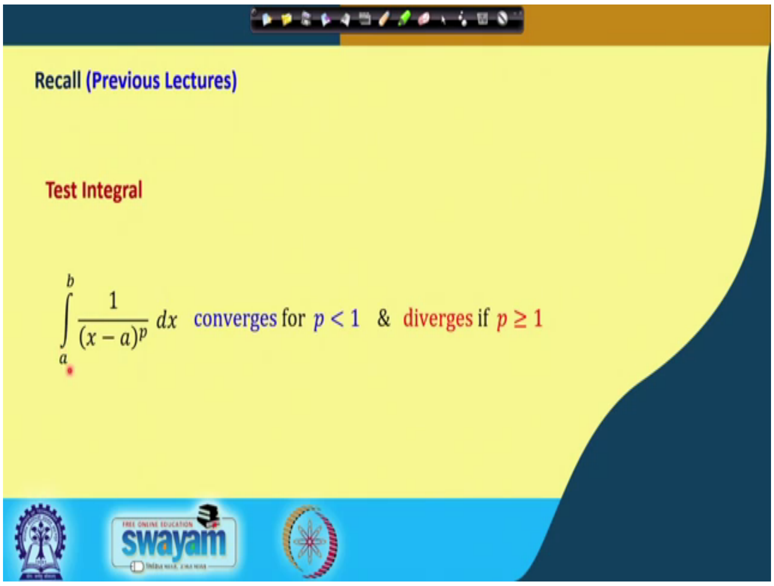

And in the last lecture, we have seen this test integral, which was a to b and 1 over x minus a power p. So, this is the improper integral of type-2, because this integrand is unbounded, when x goes to this a. So, this integral which is the test integral, and today we will use this integral to prove the convergence of other integrals. And this integral converges when p is less than 1, they strictly less than 1. And this integral diverges, if this p is greater than or equal to 1.

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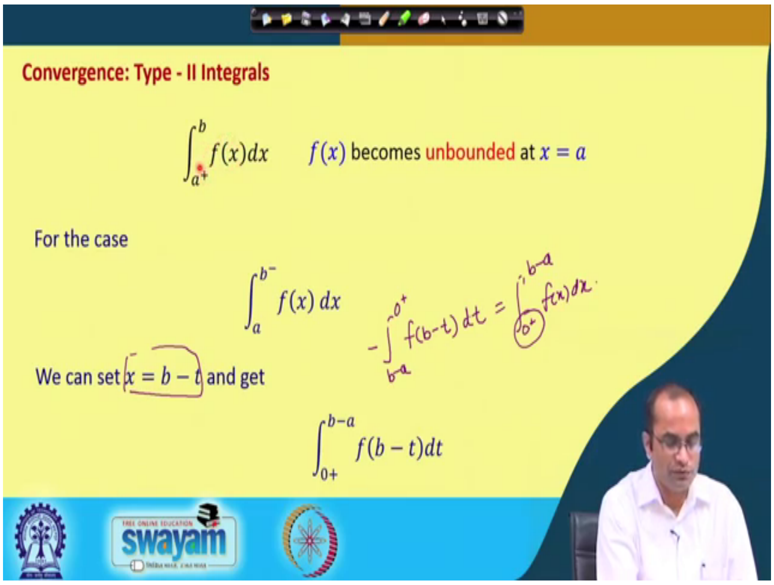

So, to start with so we have again to remind we have this type-2 integrals the improper integral, and we will be using this notation which says that this a plus this notation means, this f x becomes unbounded, when x goes to a. So, this will be the notation use, when we used here b minus for example that means this f becomes unbounded, when x goes to b.

So, for the case when we have this integral, where f x becomes unbounded as x goes to b. In that case, we will de ligand this integral similar to this type of integral by just this substitution. So, if you substitute here x is equal to b minus t, so if you substitute x is equal to b minus t, so in that case this integral will become this f and this b minus t and this d x will be minus d t, so with minus sign.

And then when x is a, so this t will become b minus a, and this when x is b minus, so this will become 0. So, this we will change the limit and we will get this 0 plus to b minus a, and then f x dx. So, this will be integral, which will be considering again that f x is unbounded, when x goes to the 0.

So, in this situation so we will be considering this integral, which is again of this nature, which we take as these standard for the discussion that f your integrand is bounded, when the argument is going to the lower limit. So, in this case here the when t is approaching to 0, so this is becoming unbounded. So, the same situation like we have in the above integral that this f x is unbounded, when x goes to a from the right hand side.

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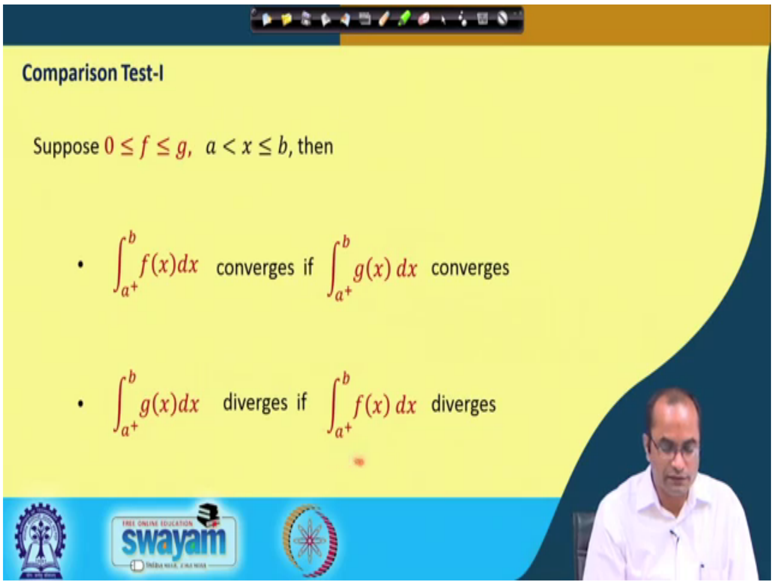

So, moving now to the next, we have the comparison test the similar to the one which we have already discussed in previous lecture for integral type-1. And in this case, again we suppose that we have this inequality for the values of the function between this a to b. So, this f is taking non-negative values, and g is taking larger values then f.

And in that case, we have that if this g the bigger one here that the g is taking large values, so if this integral a to b g x dx this converges, then naturally the integral, which where this integrand is taking the lower values, then the g this will also converge. Second term, we can also conclude out of this relation that if g integral over this g diverges, so the g will diverge if the integral the smaller one, f x dx if it diverges.

So, since this f is taking the lower values than the g. So, if this integral diverges, so naturally this integral the value is suppose to be bigger than this integral, so definitely this will also diverge. So, it is a simple comparison test, but very useful for deciding the behavior of such improper integrals.

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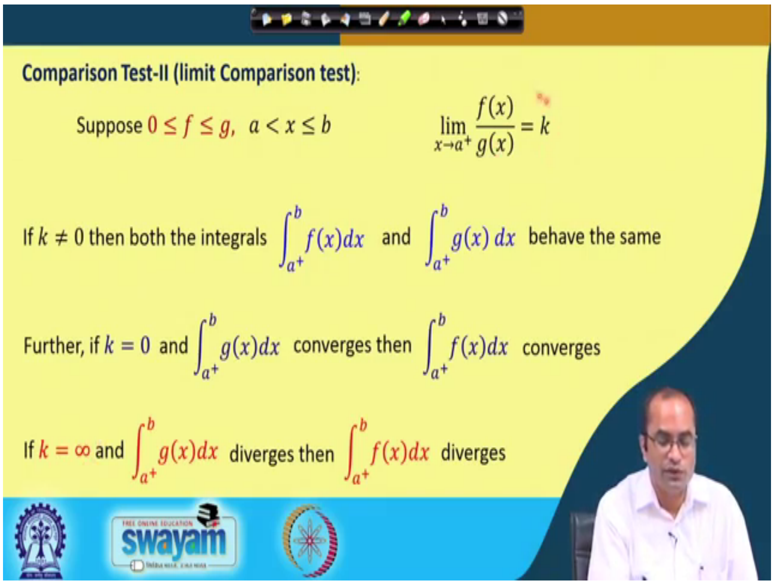

There is another test here, it is we call comparison test-2 or it is called the limit test also limit comparison test. And in this case, again we consider this in same inequality that f is taking non-negative values, and g is taking larger values then f.

So, in that case we compute this limit, which is x approaches to a plus because our problem was when x approaching to a, so we could take this ratio here f x over g x. And we have to find for what g, we are getting this conclusion here. So, we take another function based on the given function f this will be rather clear, when we discuss the examples.

So, in this case we will choose some g for given f for the other way around, and compute this limit. Suppose, this limit is coming to be k, then based on the value of k we can decide the nature of the integral for the given nature of the other integral. So, here for example if k is not equal to 0, so if you are getting this non-zero number as the limit.

So, in that case both the integrals meaning this a to b this f x dx, and this a to be g x dx will behave the same. And the reason is clear that when x goes to a, this ratio is the non-zero number that means, they behave the same as x approaches to a. And then there integral will also either both will diverge or both will converge in this case, and this is very useful.

And most of the examples exactly we use this conclusion that this j k, which we got after this limit of this ratio. And then either depending on these stress integral whether that that converges or the diverges, the other integral the behavior of the other integral we can conclude.

Further if we get for example this k to be 0, so in this case what is happening that means, this s this g is growing much faster and perhaps going to infinity than this f x, so we got this 0. So, it this seems that g is having larger values as x approaches to a. So, if this integral converges in this case this g integral, then we can tell that this integral f will also converge because of this limit here. This g will be growing faster, and that is a reason here we are getting this 0. And in that case if this g converges, the f will also converge. So, without the proof we can just by intuition, we can feel that these results hold.

Again this is the other way around that if k is 0 so sorry k is infinity, so in this case this f x is approaching to infinity is getting unbounded faster than this g function. And if this g function diverges here, so if this g diverges we can conclude that the other integral, which is a to b f x dx that also diverges. So, with this simple test we can discuss many examples, where based on the integral which we have just discuss this test integral. So, this is going to be very useful.

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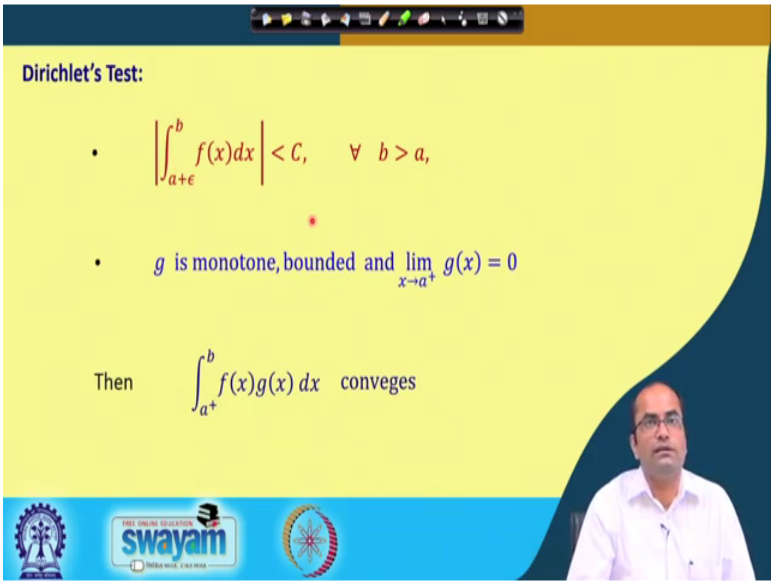

There was another Dirichlet's test for type-1 interval we have discussed, and again we have a parallel test here also, which is more or less the same that if this f integral is uniformly bounded that means for any b here greater than a, we can bound this by some constant for any epsilon positive. In that case, further if we assume that another function g is monotone and bounded, and approaching to 0 as x approaching to this a and then this Dirichlet's test concludes that this integral a to b, taking this product f x, g x also converges.

So, this Dirichlet's test is also useful, when we have this kind of product of two integrals. And one we know that this is monotone, and this is bounded also approaching to 0. And the other one, here this integral is uniformly bounded. So, in that case we can use this Dirichlet's test to prove the convergence of this, when we have in the integrand this product.

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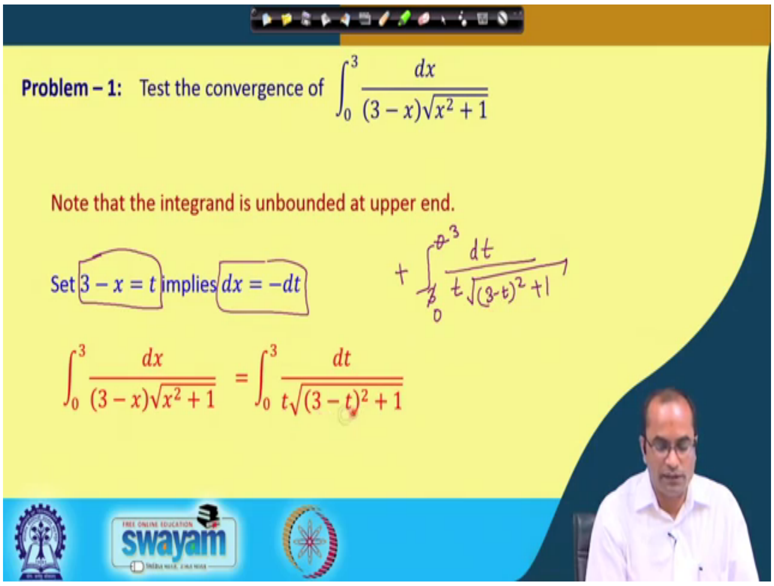

For example, we want to test the convergence of this integral 0 to 3, and dx 3 minus x a x square root x square plus 1. So, if you want to test the convergence of this integral, then we should not first that the integrand is unbounded at the upper end.

So, we are not as per the our earlier discussion that we are talking about I mean though similarly we can deal, when we have this integral is becoming unbounded at the upper bound. But, the other way we can just converted by a suitable substitution to again, so that our integrand is going to unbounded, when we are approaching to this lower boundary of the integral.

So, by substituting this 3 minus x is equal to t. So, if you substitute this one 3 minus x is equal to t, and the that will implies that dx is equal to minus dt. And this our integral will become so when x was 0, this t is 3 and when x is 3 t 0, where this dt with minus sign. And then 3 minus x is set to be t, and then we have here x square plus 1, so x is 3 minus t whole square plus 1, so that will be the integral and then making this plus here, this will be 0, and this will become 3. So, we have the integral 0 to 3 dt over t and 3 minus t square plus 1. So, having this is equal to this 0 to 3 dt over t 3 minus t square plus 1.

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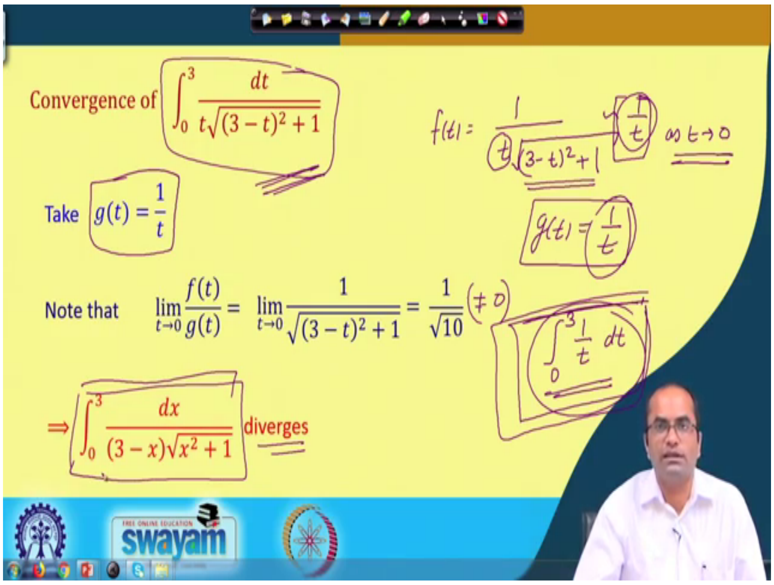

We can now talk about the convergence of this integral. And now our integrand here is getting unbounded, when this t is going to 0, so at the lower end of the integral. So, we take now for the comparison this function g t is equal to 1 over t so we take 1 over t. And the reason, why we are taking 1 over t is clear.

In this case when we have this function, so our integrand was f t here 1 over t is square root, and this 3 minus t whole square plus 1, so this was the integrand. And as t approaching to 0, so as t approaching to 0, whereas the problem this 1 over t is creating problem here there is absolutely no issues this square root, when t approaching to 0 is bounded. So, this function is getting unbounded, because of this 1 over t so that means this behavior when t approaching to 0 is said by this 1 over t. So, we will choose therefore this function g t as 1 over t, so that is the reason we have taken here, this function 1 over t.

By taking this one now we can compute this limit as t approaching to 0 f t over g t, so this f t is 1 over t and a square root this 3 minus t square plus 1. And then g t will be again 1 1 over t, so this t will get cancelled, and then in that case when t approaches to 0, we will get 1 over a square root 10, so that case this integral 0 to 3 dx over 3 minus x square root x square plus 1 this diverges, because this integral here 1 over t, when we take this integral, because we are going to now compare with this integral g t, because this tests says when limit is a non-zero number, this is a non-zero number here.

Then both the integrals both means the one is we have this 0 to 3 here 1 over t dt, so this integral. And this integral which we are testing for the convergence, and the originally this was in the form of x given by this function here. So, since this integral which we are comparing with 1 over t, this integral diverges. So, if this interval diverges, then this integral will also a diverge. And naturally the given integral this dx over 3 minus x, x square plus 1 will also diverge. So, by simple this comparison test, we are able to tell without explicitly computing the value of the integral that this integral diverges.

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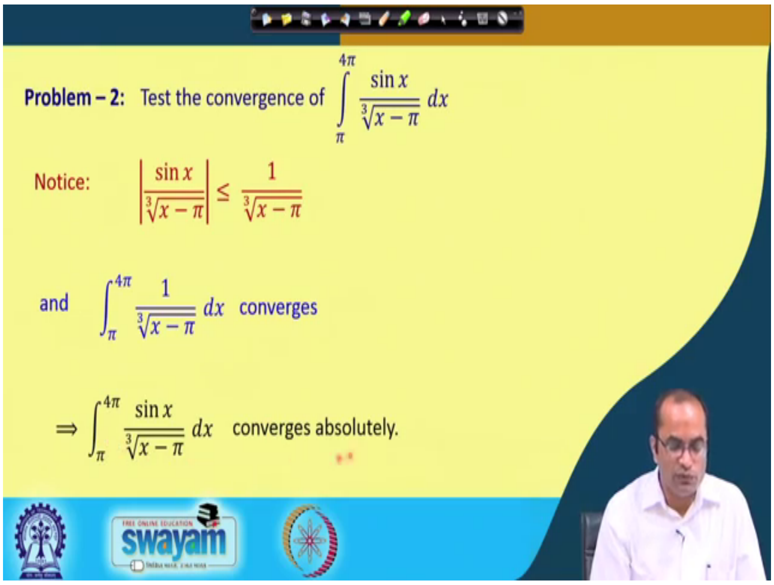

Now, we will test the convergence of this integral, where we have pi to 4 pi and sin x over a cube root x minus pi dx. So, now we have this problem again, when x is approaching to infinity, this integrand is approaching to is getting unbounded. And now we will discuss the convergence here. So, here it is rather easy to see that if we take given the absolute value of the integrand, and we have discussed already in the last lecture, which is called the absolute convergence.

So, if you if we take the absolute value of this integrand, and even though that integral converges we call that the integral converges absolutely. So, in this case even we take this absolute value of the integrand, and this is bounded by 1 over because this sin x is bounded by 1. So, this integral this absolute value is bounded by 1 over the cube root of x minus pi.

And in this case when we have this so and we know about from the test integral, how this integral behaves. So, this integral 0 to pi 1 over this cube root x minus pi dx, this converges because the power is 1 by 3 which is less than 1. So, this integral converges for any power here x minus p, x minus pi power any power here less than 1 this integral converges. So, here we have this power 1 by 3, so this converges.

And in that case we have that this integral pi to 4 pi sin x over this converges absolutely. So, we have the absolute convergence here, and this is that means, definitely this integral converges, because we have the absolute convergence. So, absolute convergence implies the convergence of the integral.

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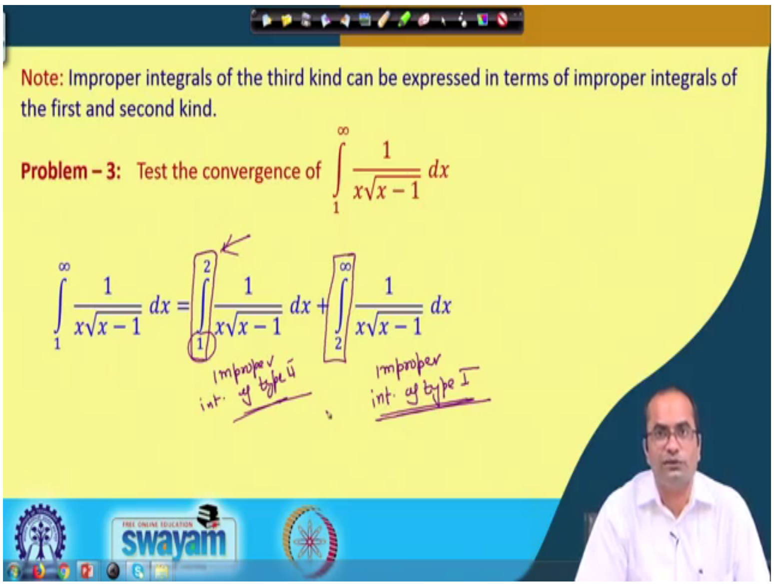

Now, if we have the improper integrals of 3rd kind, so you just to recall what was the improper integral of third kind, it was the mixture of the integral of kind 1 and kind 2.

So, we have for example we have this integral 1 to infinity, 1 over x square root x minus 1 dx. So, we have both the cases here, we do see this infinity in the limit of the integral. And also when x is approaching to 1, this integrand is getting unbounded. So, we have the integral of type-1 as well as type-2.

So, such improper integrals of we can express in terms improper integrals of the first and the second kind, and then we can basically evaluate such integrals or we can talk about the convergence of such integrals. So, in this case, we have an integral is 1 to infinity 1 over x square root x minus 1 dx, and this we can break into two integrals. So, in the first one we have taken the limit from 1 to 1 to 2 so this limit 1 to 2.

And in the second case, then we have taken from 2 to infinity. Now, what happen? In the first integral, we have no infinity. So, in that case, but this integrand is getting unbounded, when x is approaching to 1. So, this is the, a improper integral of improper integral of type of type-2, which we have discussed.

And here this is the infinity, other than this there is no problem the function is bounded. So, in whole range from 2 to infinity, and this is the improper integral of so improper integral of type-1. So, we have break this integer into two parts. The first one is the improper integral of type-2, the second one is then improper integral of type-1, and then we can discuss separately the convergence of each.

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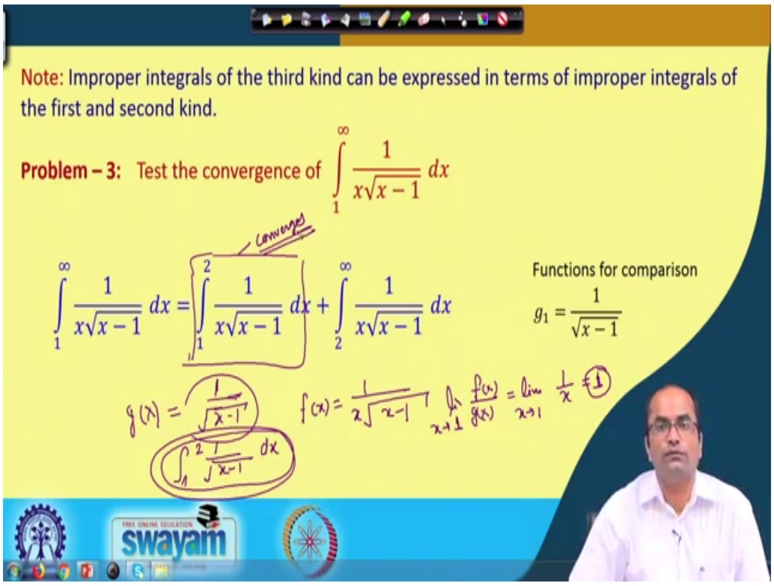

So, for the convergence, we will use this comparison test and for the first integral here, if we discussed first. So, we will take this function 1 over a square root x minus 1, and the reason is clear. If we take a look at this integral, here the first integral in this case, we have this integrand 1 over x square root x minus 1, which is getting unbounded, when x is approaching to 1. So, this factor here 1 over this is square root x minus 1 is creating this unboundedness in the function as x approaching to 1.

So, we will choose we will compare with the function, which is we have taken this here g x or g 1 x as 1 over square root x minus 1. If we take the ratio of the 2 here, so f was our integrand. So, here the f x in the first integral was 1 over x and x minus 1. In the second integral, the g x is 1 over square root x minus 1 and if we take this limit f x over g x, and then the take the limit x approaching to 1.

In this case what will happen now, so limit x approaching to 1. And this f x over g x, so this 1 over x will survive, the other one will cancel out 1 over a square root x minus 1, and as approaches to 1 this is 1. So, we are getting a non-zero number here as the limit of this ratio that means, this integral the given integral and the integral of this g.

So, integral 1 to 2 and this 1 over a square root x minus 1, they both will behave the same. And we know from the test integral that this integral 1 over a square root x minus 1 d x and 1 to 2 this integral converges. So, hence the given integral this also converges. So, this integral converges, we have seen from the comparison tests that this integral of type-2 converges.

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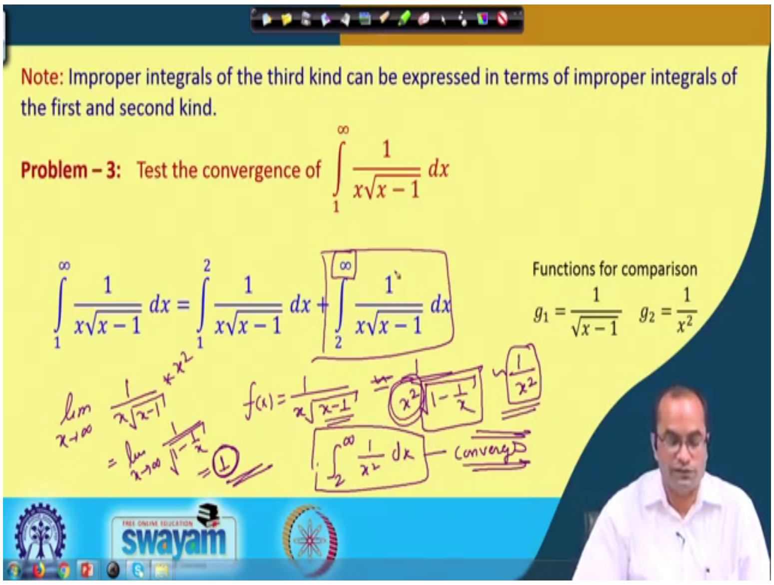

Now, we will look at the second integral. So, for the second integral, we will choose the function 1 over x square the reason is again clear, because here the function the problem is because of this infinity. So, we will check the behavior of this function as x approaching to infinity, and from there we will take the function g for the comparison.

So, if we take this f x here, which was the integral. So, 1 over this x and the square root x minus 1, so it behaves when x approaching to infinity so, this is like first I will rewrite this 1 over this x square, and here I will take 1 over 1 minus x, so I have taken this x 1 to this one.

And now if we look at the behavior as approaching to infinity, so this term here is not creating trouble, because x goes to infinity this is 1. And so we have this x square, so the behavior of this function is like 1 over x square as x approaching to infinity. So, we got the function here g as 1 over x square, so taking this g as 1 over x square.

And if we get this limit for the comparison test, so the limit x goes to infinity. And this f x over g x, so the f x was 1 over this x, and this square root x minus 1, and then this g x again 1 over x square, so we have x square there. So, this is limit x goes to infinity, and again we will take this common x, so x square we will get cancel. And we will have 1 minus 1 over x and the a square root. So, as x goes to infinity, this will approach to 1.

So, we have this limit as 1, and in this case again for the same reason both the integrals will behave the same. So, one this given integral here, and the another one our test integral, what is the test integral? Test in a 2 to infinity with this g function 1 over x square dx. And we know already that the test integral converges, so this converges so this integral converges. And hence the given integral the second integral, which was the type-1 integral that also converges.

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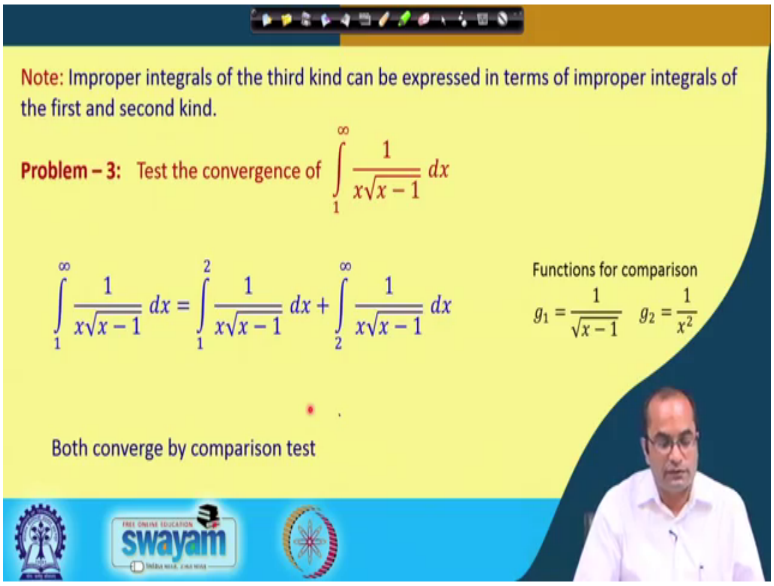

So, what we have observed now here that both the integrals on the right hand side, they both converges by this comparison test.

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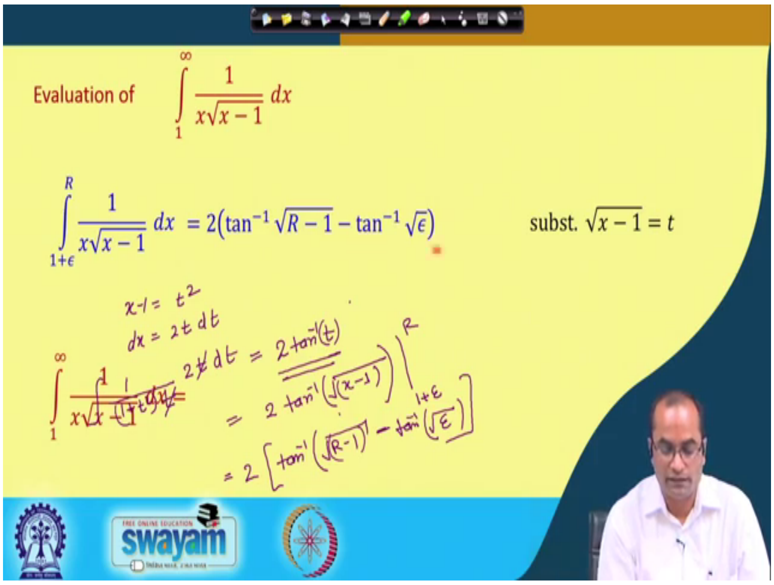

Indeed now in this case, we can also evaluate this function exactly using the same idea by breaking this integral into two parts, where the first integral will be a integral of type-2, the second will be of integral type-1. And then we can evaluate each one or we can consider now this integral the same idea, that we have avoided here this one by taking this 1 plus epsilon, and the upper limit which was infinity, we have replaced by a number R a large number R, and taking this 1 over x and this x minus 1 dx.

Now, we will evaluate first this integral, because it is a proper integral we have no problem at the lower end and also at the upper end. So, here we will substitute this square root x minus 1 to t, and then we can easily evaluate this integral. So, by doing this substitution, so we have a substituted this x minus 1 is equal to t square that means, we have this dx is equal to 2 t and dt.

So, in this case this integral so if we talk about this in definite integral, so 1 over the x is replaced by 1 plus t square now, and this is square root x minus 1 is replaced by t, and this dx by 2 t into dt. So, this t gets cancelled, and this integral becomes just 2 over 1 plus t square dt, which we can integrate. So, this is 2 and tan inverse t, so that is a value of the integral.

And now we can put that limit back. So, first this x, so here 2 tan inverse and the t was a square root x minus 1 x minus 1. So, and now the limit from 1 plus epsilon to R, so putting these limits here now, we got 2 and this tan inverse the square root R minus 1, and then minus this tan inverse, and this x is replaced by 1 plus epsilon, so 1 gets cancels and we get a square root of epsilon. So, this is the value of the integral the given integral, which is given here the 2 times the tan inverse R minus 1 with the square root and minus tan inverse epsilon.

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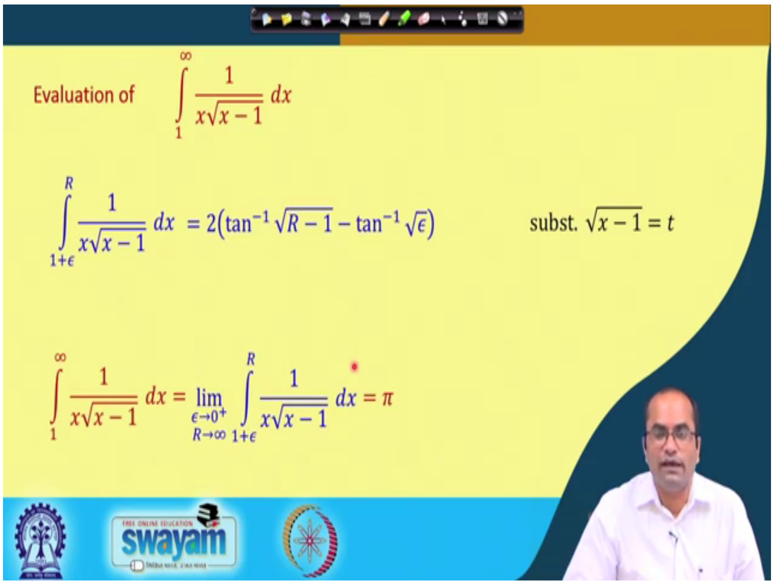

So, with this now we can approach to the given integral. So, in this case, this integral here 1 to infinity 1 over x and a square root x minus 1 dx will get when we take the limit this epsilon to 0 and R 2 infinity. So, in this integral the value of this integral, we will pass the limit as R approaching to infinity and epsilon approaches to 0.

So, when R approaches to infinity, you note that here R approaches to infinity tan inverse this infinity, which will become pi by 2. And minus this tan inverse, when this epsilon approaches to 0, so tan inverse 0 that will become 0. So, we have because of this pi by 2 and 2 times, so this value will be just pi.

So, again not, so this is this is here tan inverse tan inverse pi by tan inverse infinity, so that will be pi by 2 and this is minus 0 and then we have here 2 times. So, therefore, we got this pi here. So, the value of this integral is pi, and we have seen before as well without evaluating the value that this is a convergent integral.

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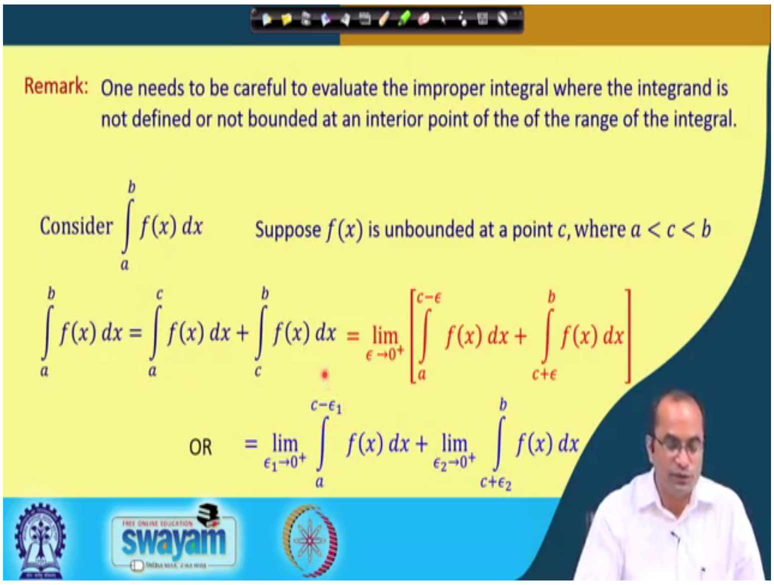

And just a remark here that one needs to be careful to evaluate the improper integrals where the integrand is not defined or it is not bounded at an interior point of the range of the integral. So, if we have a point not the end point, if you have a point in the middle where the integral is getting is integrand is unbounded or it is not defined, we have to be very careful for the evaluation.

So, suppose we have this integral a to b f x dx. And further we let that f x is unbounded at a point c, where this c is a point between a and b. So, in this case this f x is getting unbounded somewhere at the point c, which is in the range of this integral. And then we have to break this at that point, so a to c fx dx, and then c to b f x dx.

Now, we have these two integrals of type-2 integral. So, here this is getting unbounded, when x approaching to c. And this is also getting unbounded, when x approaching to c. So, there are two ways now, one can think of getting this limit one is like we introduce this epsilon, and epsilon we take the limit as it goes to zero. And we have a to c minus epsilon, and here we have then c plus epsilon to b. So, at these two places, we have avoided that point c by c minus epsilon and c plus epsilon, we have taken the same epsilon at both the places.

In the second case, which is the correct one for the evaluation basically, we take the epsilon one here for the first integral, and because these two are the separate improper integrals, so we should deal differently. So, here we have the limit epsilon 1 to 0, a to c minus epsilon, and plus this epsilon to 0, when c plus epsilon 2 to b. So, we have now considered, these two integrals differently here we have introduced one epsilon, and then avoided these point there.

So, in the case when we have convergence integrals, so there is no problem whether you take this one or this one, this will come of the same. So, but in other cases we have the problem here, because this might be a divergent integral and this might be the divergent integral and in that case the value will differ.

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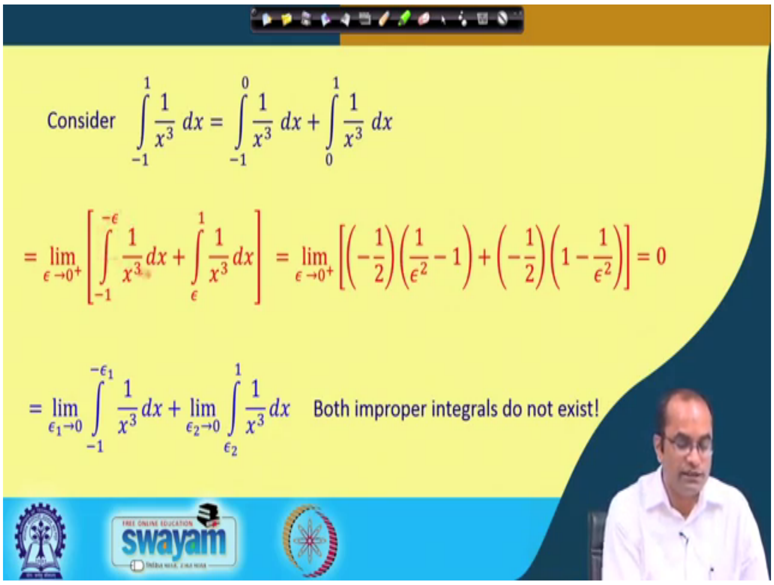

So, for instance we consider the simple problem here minus 1 to 1, 1 over x cube this dx, and this is equal to minus 1 to 0, 1 over x cube this d x plus 0 to 1 1 over x cube dx. So, if we take these two integrals, in that case if we use this first approach by introducing only one epsilon here. So, what will happen, when we compute this 1 over x cube, which will be minus 1 over 2 x square and putting this limit here minus 1 to epsilon, we will be getting this here for the second integral again, we will be getting 1 minus 1 over epsilon.

So, now in these will cancel out, and this limit will be 0 basically. But, what will happen if evaluate separately the first integral from minus 1 to minus epsilon, and then epsilon 2 to 1 in that case none of the integrals here, because we will get exactly this here with epsilon 1. And when epsilon 1 goes to 0, this will be unbounded this will go to infinity. Similarly, this is also not in convergent integral, this also diverges and that is the reason, we cannot evaluate this here, because both the integrals appearing at this place they diverges and they do not exist. But, if we do this mistake here by taking the epsilon at both the places, then the value is coming to be 0.

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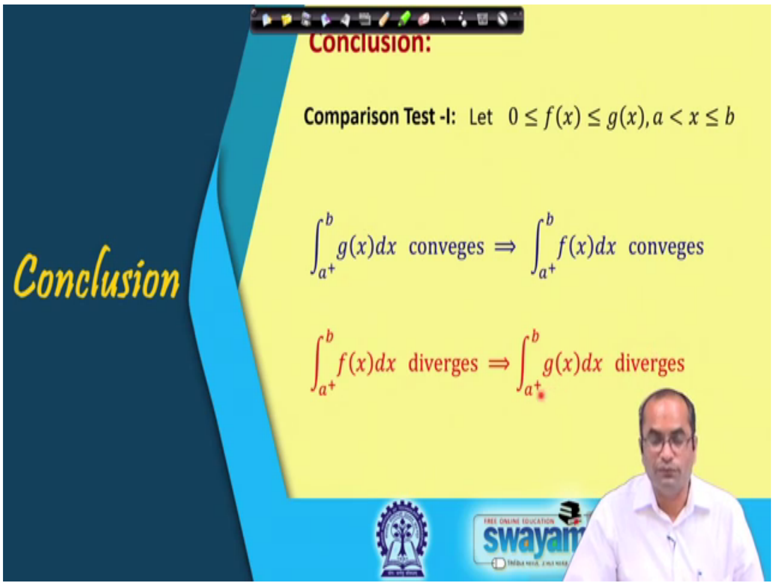

So, coming to the conclusion. So, we have these two important comparison test, so which was one here, where we take this f x greater than 0, and the g x greater than equal to f x. And then, we notice that if this g converges the bigger one the natural, this is smaller one will also converge. The other way around if this smaller one diverges, we can conclude that this the larger one will also diverge, so we have this comparison test.

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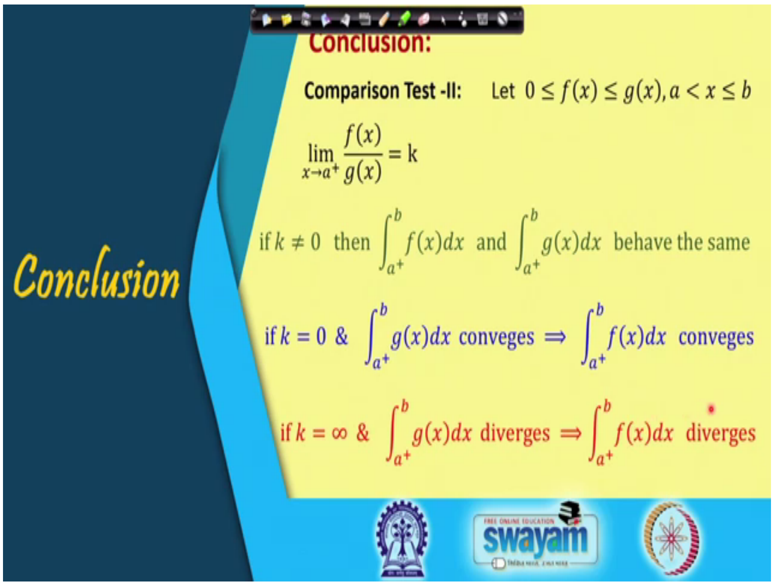

And another this comparison test-2, which was also very important and very useful where we of take a another function g and compute this limit, and they based on this number k, we can conclude the convergence. So, if k is not equal to 0, both the integrals behave the same. And if k is equal to 0, the a if this g integral converges, the f will also converge. And if k is infinity, which implies that g would diverges, so the f will also diverge. So, this was the comparison test-2.

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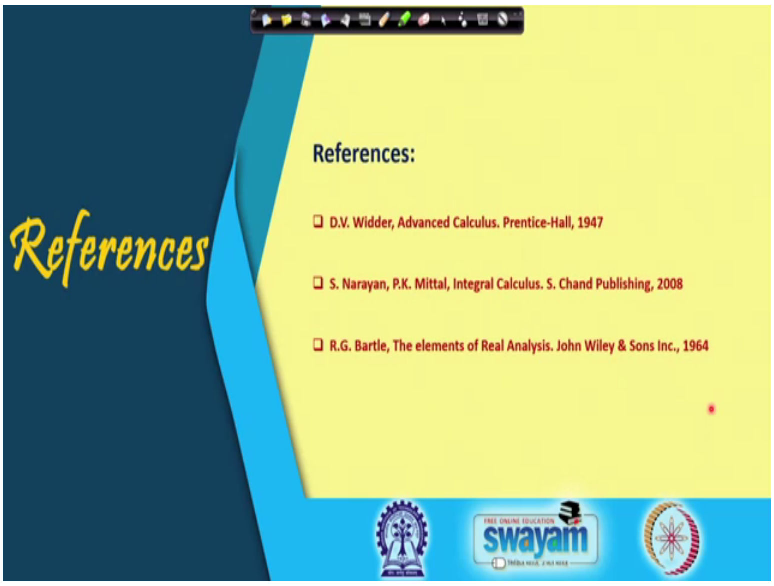

And these are the references we have used for preparing the lectures.

And thank you very much.