Engineering Mathematics – I
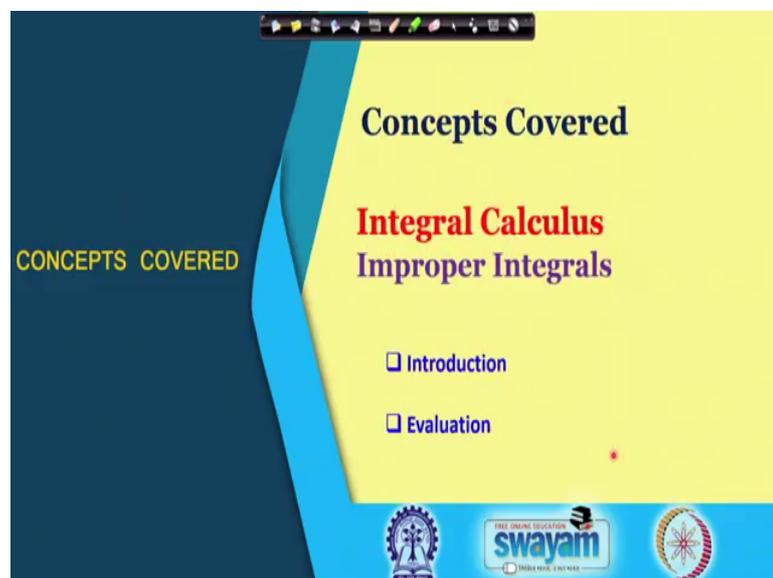**Prof. Jitendra Kumar**
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Lecture – 21
Improper Integrals

So, welcome to the lectures on Engineering Mathematics – I and this is lecture number 21 and today, we will talk about the integral calculus and in particular Improper Integrals.

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So, we will introduce first the improper integrals and then how to evaluate such integrals that will be the topic of today's lecture.

So, here there are proper integrals which we usually studies so, those integrals. So, for example, the integral a to b f x dx is proper. If the range here a to b of this integral is finite and the integrand so, the f x is the integrand here and that is bounded in this range a to b for x. So, if this f x is bounded and a and b both are finite so, the range of the integral is finite in that case we call that this integral is proper. But, now we will discuss today what are the improper integrals.

So, improper integral the integral 0 to b a to b f x dx is improper. If this a is for example, minus infinity and or b is plus infinity so, one of them is basically infinity that a is minus infinity or b is plus infinity or both, a is minus infinity and b is infinity and this f x the integrand here is unbounded at some point in the range of this x from a to b. In that case we call this integral improper integral of first kind.

The second, if this f x is unbounded at one or more points in this range a to b then we call improper integral of second kind. So, in the first kind the limits either a or b is infinity or both are infinity minus infinity to plus infinity, but here the f x is bounded in the integral of first kind whereas, in the integral of the second kind we assume that this f x is unbounded between this a and b and these limits here the range of the integral is finite from this a to b.

The second type or the third kind of improper integral is when we mix these two types of integrals. So, the range one of the range here either a or this b is infinity or both are

infinity as well as the f x the integrand here is also unbounded at some point in the within the range of this x then we call that this is the third kind of improper integral.

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So, the example for proper integral: so, for example, we have here 0 to 2 in the square root x square plus 1 dx. So, here the integrand which is x square root x square plus 1 is finite or bounded in the range of this x from 0 to 2 and also the range the 0 and the 2 that is also finite. So, we do not see here infinity and also this f x, the integrand is also bounded. So, in this case we call such integrals proper integral or for example, we have 0 to 1 sin x over x dx. So, in this case one might think that here the x is in the denominator and the range of this x is from 0 to infinity then the integrand may become unbounded.

But, this is not the case because if we consider the limit here so, the limit as x goes to x goes to 0 and this sin x over x. So, this limit is finite, the limit is 1, we know that this limit is 1. So, here the limiting value of this integrand as this x goes to 0 is infinite, in that case also we will call this integral as a proper integral because this is not unbounded at any point in the range of this 0 to 1 and the limits are also finite the range is finite. So, this integral is also proper integral.

Coming to the improper integrals: so, here we have for example, this 0 to infinity cos x dx. So, this is the improper integral of a kind one or the first kind because here in the limit we do see a infinity, but the integrand at any point is bounded. So, this is the integral of first kind.

Second, this 0 to 1 1 over x minus 1 dx; so, in this case the range is finite, but the integrand here as x approaches to 1, this is becoming unbounded. So, in this case the integrand is becoming unbounded as x approaching to this upper limit a and then this is the improper integral of a second kind. This is here the integral of third kind or the mixed kind because the limit is also unbounded. So, here we have this infinity in the upper range and then this integrand which is 1 over 1 minus x square whole square is also going to infinity as x is approaching to 1 and 1 is in the range of the integral. So, in this case the integrand is also becoming infinity and the range is also not finite. So, this is the integrand of improper integral of third kind.

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So, how to evaluate such improper integrals, for first we will discuss for the first kind. So, for example, we have this a to b f x dx and suppose that this b is going to infinity. So, this is the integral of first kind and we assume here that this b is approaching to infinity. So, in this case what we will consider, we will consider the integral a to R. So, this b which is infinity we have replaced by this R and then we are taking this R to infinity.

So, in this case here we have like this infinity. So, this is the integral of first kind and to evaluate this what we will do we will replace that infinity by R this number and then later on after evaluating this integral here a to R f x dx and then we will take this limit and if this limit exists we call that this is the value of this integral or the integral

converges and if this limit does not exist then we call the limit that the integral does not exist or the integral diverges.

Well, so, now in this case we may have the situation that this instead of showing the earlier integral here this b was infinity and now we have considered that this the lower range is minus infinity so, again we have the improper integral of first kind. So, in this case we will use the same idea here the limit we will take over R to infinity and this infinity will be replaced by minus R. So, now, we will evaluate this integral first from minus R to b f x dx and then we will take the limit as R tending to infinity after the evaluation of this integral.

So, if we have this again the first kind integral, but we have the both the limits here minus infinity and this plus infinity so, in this case what we will do we will break this integral into two parts. So, one we will take this limit as R 1 to infinity. So, here we have taken this minus R 1 for this infinity we have replaced by R 1 and for this R 2 we have replaced by this is replaced by this plus infinity there.

And, we have taken this R 1 to infinity and R 2 to infinity so, again we have this integral here minus infinity to plus infinity and if these limits here on the right hand side these exists then we call that this limit is convergent or this integral is convergent otherwise this is a divergent integral or the integral does not exist. Or, we can combine these two like limit R 1 to infinity R 2 to infinity and we take from minus R 1 to R 2 f x dx this is the same.

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So, now we will do how to evaluate improper integrals of say first kind. So, this is the example here we have the 2x square and over x 4 minus 1 dx 2 to infinity. So, in this case the tree case that we will this we will remove this infinity and we will replace that with the R. So, we have this integral equal to the limit R to infinity and this 2 to R and 2x square over x 4 minus 1 dx. This is equal to limit R to infinity this we will do the partial fractions now. So, this will be x square minus 1 and x square plus 1; so, 1 over x square minus 1 and then plus 1 over x square plus 1. So, this will be exactly 2x square over x 4 minus 1 so dx.

And, then again we will do this partial fractions there. So, we have the limit R to infinity this 2 to R and this will be 1 by 2 and this is 1 over x minus 1 minus 1 over x plus 1. So, this is the partial fraction of 1 over x square minus 1 and then we have this plus 1 over x square plus 1 and everything over this dx term.

So, here we have the limit R to infinity and this is 0 to sorry 2 to R. So, we have three integrals now. So, one is the 1 by 2 and this 1 over x minus 1 dx, then we have minus sin over x plus 1 dx the integral this 2 to R again and then we have the integral 2 to R and 1 over x square plus 1 dx. So, the first 1 let it be limit at R tending to infinity and we have here this half this will be ln x minus 1 and this will be minus x plus 1. So, this will be like ln the logarithmic of a x minus 1 over x plus 1 combining these two the first two and then here we have the tan inverse x that is the integral the limit 2 to R.

So, now we will compute to this limit here. So, that will be half and we are putting so limit at the end; so, R to infinity we have this half and we have this ln upper limit; so, R minus 1, R minus 1 over R plus 1 and minus this 2. So, minus 2; so, minus here 2 minus 1 1 and then here we have 3. So, we have 1 by 2 and this ln. So, with minus sign because of this lower limits.

So, minus half and then 2 minus 1 so, ln 1 and then by 3; so, ln 1 by 3 and then we have here tan inverse tan inverse R and minus tan inverse 2. So, and then everything we have this R to infinity. So, 1 by 2 and this will be a ln 1 minus 1 over R over 1 plus 1 over R this still the limit, R to infinity. Here we have this we can write down a plus half ln 3 and then we have here plus tan inverse R and then minus tan inverse 2. So, taking this limit now we have the half and then here ln this R tending to infinity. So, this is 0, this is also going to 0, we have ln 1 that is 0. So, half into 0 plus this half ln 3 and then tan inverse as R tending to infinity will be pi by 2 and minus tan inverse 2.

So, this is the value, this is 0 here, we have half ln 3 and this pi by 2 and then minus this tan inverse 2.

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So, this is the value of the integral which we have computed here pi by 2 minus tan inverse 2 and this half ln 3. So, this integral exists and this is the value of this integral improper integral. The second for example, we have 0 to infinity and sin pi dx. So, in this case also using the same tricks so we have the limit R to infinity and then 0 to R and we

have sin x dx this integral will be R to infinity sin x will become minus cos x and then we have the limit 0 to R.

So, this will be limit R to infinity and here we will have this 1 minus cos R and when we take the limit R to infinity this for this cos R this limit does not exist. So, therefore, in this case the second case the limit does not exist and we have precisely this limit 1 minus cos R and the limit does not exist or the integral does not converge in this case. So, the limit does not exist. So, in the first case this improper integral we call it is convergent and the values given by this expression here. In the second case this integral the improper integral does not converge because we cannot evaluate this integral, ok.

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So, the evaluation of the improper integral of the second kind we will learn are the integral of the kind a to b f x dx where f x is unbounded at some point in the range a to b. So, here this suppose this f x is infinity as x tending to b; so, the upper limit when x is approaching to upper limit this f x is converging to is going to 0 is becoming unbounded and for such type of integrals when this f x is approaching to infinity as extending to b.

We will evaluate by taking by introducing this epsilon here and so, here we have introduced this epsilon and subtracted from the b. So, the problem was at b. So, we have voided now by b minus epsilon because the f x was tending to infinity when x was tending to b. So, we have avoided now this b making this b minus epsilon and now this is nice integral, the proper integral which we will evaluate using the techniques developed

for the evaluation of the integral and later on we will take the limit epsilon tending to 0 from the right side.

So, here we have this integral and in the second case when f x goes to infinity as x goes to a you will use the same idea that now we will use here a plus epsilon. So, again this a where the problem was there in the integral is avoided and later on we will take this epsilon approaches to 0 from the positive side to evaluate this improper integral.

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So, evaluation of improper integral of second kind when we have a some other point as x approaching to c c is a point in a b and this f x is becoming unbounded so, this is the improper integral of the second kind and in this case again we will use the trick that the integral a to c and c to b we will break and since the problem was at c so, we have avoided by subtracting epsilon from c. So, we have a to c minus epsilon f x dx and in the second case the again the integral was c to b, but we have the problem at c, so, we have removed this by taking c plus epsilon and then epsilon are tending 0 from the right side.

So, if you have this integral f x goes to infinity at both the ends like x goes to a and also x goes to b. So, this is another for the second kind of integral. So, in this case we can evaluate this a to b f x dx by introducing this a plus epsilon 1 and b minus epsilon 2 and then taking these limit epsilon approaches to 0 plus and epsilon 2 approaches to 0 minus.

So, now, we will take some example here where we can evaluate just again a remarks. So, in this case when we have this f x is approaching to infinity at both the ends. We can also break this integral at some middle at some other point here like a to c and then c to b and then we can handle exactly the integrals we have handled before. So, we will avoid we have two different integrals and c dot this 1 and the first one we will we will handle with this a plus epsilon and the second one we will handle with that point to this b minus epsilon similarly as we have done above.

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So, here how to evaluate such improper integrals of the second kind; so, for example, we have this dx over a square root 1 minus x. So, here when x approaching to 1 this is becoming unbounded our integrand is becoming unbounded. So, in this case what you will do we will take the limit epsilon to 0 from the right side here and 0 to 1 minus epsilon.

So, the point where the function was becoming unbounded we have replaced by 1 minus epsilon. So, we are not going from 0 to 1, but we are going from 0 to 1 minus epsilon where we have the nice integral and later on we will substitute we will take that limit there. So, what is this integral value here? So, we have this integral 0 to 1 minus epsilon and this is 1 minus x power minus half dx which if we integrate we will have minus 1 because of this because of this minus sign minus half plus 1. So, that will be half.

So, we have 1 minus x power 1 by 2 or the square root and then we have this limit and this 1 by 2 will be also in the in the denominator. So, we have here the minus 2 and the square root here of 1 minus x and then we have this limit 0 to 1 minus epsilon; so, 0 to 1 minus epsilon. So, the value here of this integral when we put this minus 1 minus epsilon we have 1 minus and 1 minus epsilon and minus square root 1, that is 1; so, 1 minus 0. So, here we have minus 2 and this will be epsilon and minus 1. So, we have square root epsilon with minus 2 and then we have this minus 1 there.

So, now, we will take the limit. So, taking this limit here limit epsilon goes to 0 with minus 2 and square root epsilon plus 2 here. So, this epsilon goes to 0 then we have this limit as 2. So, the limit of this after taking the integral then, we are taking the limit the value is coming to be 2 in this case.

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So, that is the value of this integral this integral convergence and the value is 2.

The second example we take dx over 2x minus x square and in this case we will do again the same trick that this limit epsilon 1 goes to 0 and we have avoided this 0 by introducing 0 plus epsilon 1. And, we have taken some point in the range so 1 and then from 1 to 2 minus epsilon 2 where epsilon 2 goes to 0. So, we have avoided also this a point where the function was becoming unbounded. Now, in both the cases we have to integrate this and then later on we will supply this limit as epsilon 1 and this epsilon 2 goes to 0. So, let us look at this integral here we have basically this integral dx over x and 2 minus x which we can write down as 1 over 2 minus x and then we have plus 1 over x.

So, this is 1 by 2 is we have to adjust here because we will get this 2 there. So, we have 1 by 2 1 1 by 2 minus x plus 1 over x and this integral there over dx which we can integrate. So, we will get this with minus sign as ln 2 minus x and plus this ln x which is basically this half and this length x over 2 minus x that is the integral and now we will take the first integral. For example, here where the limit as epsilon 1 goes to 0 from the positive side and this values we will write down first so, above here 1. So, 1 minus ln will be 0 when we put the lower one. So, we will get half and ln epsilon 1 over 2 minus epsilon 1 the limit epsilon goes to 0 plus.

The second 1 we will have this limit epsilon 2 goes to 0 plus for the second integral and we will put 2 minus epsilon first. So, here we have half ln and for x we are writing 2

minus epsilon 2 over 2 minus for x again 2 minus epsilon 2 and then we have this 1 there that will be again 0. So, we have these 2 values 2 integrals now. So, in the first 1 when this epsilon goes to 0 and and there is 2 minus epsilon one. So, this is going like ln and 0 from positive side. So, we have this infinity to them there in fact, minus infinity then epsilon goes to 0. Here when we take epsilon 2 goes to 0. So, we have this is going to 0 and then so, 2 divided by here epsilon 2. So, epsilon goes to 0 this will be becoming again the infinity. So, this integral does not converge because we are getting the infinity here.

Well, so that is true. So, we have 2 minus x and this is infinity. So, here going to infinity this is also going to infinity. So, our integral is becoming infinity and this infinity plus infinity. So, this is not convergent.

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So, in this case we will we get this value I mean this integral does not converge because, this is converging to going to infinity the value and this integral diverges.

So, here we also use in further lectures about this test integrals because in the next lectures we will learn about how to how to test whether this converge this integral converges or it diverges without evaluating them. So, there we will be using some this test integral. This is for test integral – 1, this will be used there.

So, we have a to R this later on we will as take the limit R to infinity. So, we have this integral a to R 1 over x x power p dx which we can easily evaluate, this a is positive number; so, here 1 over x minus b. So, when we integrate this. So, we have 1 over x and the p and this minus 1 will come. So, x power minus p plus 1 and then so, minus p plus 1, so, 1 minus p will also comes. So, this is the integral then a to R, but this is like taking p is not equal to 1 because when we take p is equal to 1 we will get this 1 over x which is the logarithmic here of x and then we will take the limit.

So, there will be two possibilities when p is equal to 1, then this will be the integral and then p is equal to 1, this will be the integral and then we can substitute the upper limit here R power p minus 1 and we can substitute the lower limit as a and we will get this integral here. So, for p is equal to 1 w have ln R and there is minus l n a which is ln R over a and in the second case when p is not equal to 1 we will get 1 over 1 minus p and this 1 over x power p minus 1 and here again this R so, first 1 over R p minus 1 we will get and then minus 1 over a power p minus 1.

So, these are the integrals integral value for this a to R and now we will see here what will happen to when we take this R to infinity. So, basically we have a interest about test integral is a to infinity 1 over x power p dx. We have already evaluated this now we will put the limits.

So, if R tending to infinity this will become a can this ln infinity. So, the infinity p is equal to 1 again we will get infinity and in this case also when this p is for example greater than 1. So, p is greater than 1 this R power will be positive here and when R tending to infinity so, this will become 0 and then we will have this finite value, but when p is less than 1, when p is less than 1 then this R will he having positive power in the numerator and when R tending to infinity so, this will become infinity this term and then the integral will become infinity.

So, p we have the two values here when p is less than 1 or equal to 1 the value of this integral is infinity, the integral divergence and when p is greater than 1 when p is greater than 1 in that case this term will vanish and we have 1 over this a power p minus 1 and this minus sign we have adjusted here to have p minus 1. So, in that case the value is 1 over p minus 1 and 1 over a power p minus 1 for p greater than 1.

We have one more integral which will be used for the test integral. This is called test integral – II and in this case we consider this integral a plus epsilon to b. So, here we have that this as x approaching to a this is becoming infinite so, unbounded so, we have the second kind integral. So, a plus epsilon to b we will evaluate this and again here the same idea. We have 1 over x minus a power p which we can integrate. So, there will be two cases again, when p is equal to 1 we will have this lnx minus a which will give us this is ln b minus a minus ln epsilon and when p is not equal to 1 then you will integrate this with this will be 1 over x minus a power p minus 1 and which will result into this expression here with 1 over 1 minus p.

Now, we will again discuss the case that what will happen when epsilon goes to 0. So, it is clear when epsilon goes to 0, in this case here when epsilon goes to 0 you have minus ln epsilon and epsilon is going to 0 so, this will become infinity. So, p is equal to 1 again this integral is not converging and here we have the situation when epsilon goes to 0. So, this so, p is greater than 1 then this will become the infinity here if p is greater than one1.

So, this will become infinity and here also p greater than 1. So, this is also minus this is minus. So, we will have a this integral is converging to plus infinity and when p is less than 1 so, this will epsilon go into the numerator and when epsilon goes to 0. So, this term will also go to 0. So, in that case this integral will exist. So, we have the situation when p is greater than or equal to 1 in that case this integral is converge is not

converging, it is diverging the value is going to plus infinity and when this p is less than 1, in that case we have the value finite value of this integral which is given by 1 over 1 minus p and 1 over b minus a power p minus 1. So, again this integral will be used as a test integral in further lectures and we will use this result that p greater than equal to 1 this is infinity and this integral converges when p is less than 1.

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So, today we have discussed the improper integral of this kind and there were two are possibilities that a maybe minus infinity b is and over plus infinity and f x is bounded this was the integral of first kind. We have also seen that there might be a situation that f x is unbounded at one or more points in this interval a to b a is less than equal to a x is less than equal to b. So, in that case also this is a improper integral of a second kind and they we have seen how to evaluate those integrals basically using that limit. So, if it is a of first kind then this infinity whether it is in the lower range or the upper one you will replace by a finite number R and then we will evaluate the integral later on we will take the limit as R tending to infinity.

In the second case at the point where the function is unbounded that we have to avoid by introducing a small number epsilon which later on we will move to 0 will go to 0 in the limiting scenario. So, we can evaluate those integrals, but in the next lecture we will see that how to how to find without evaluating whether the integral converges or it diverges.

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So, here these are the references we have used to prepare these lectures and.

Thank you very much.