Introduction to Abstract and Linear Algebra
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Lecture – 13
Subgroup

Ok. So, we are talking about group if we have seen the order of an element, now we will talk about a cyclic group, but before that let us start the concept of subgroup. So, how to define the subgroups?

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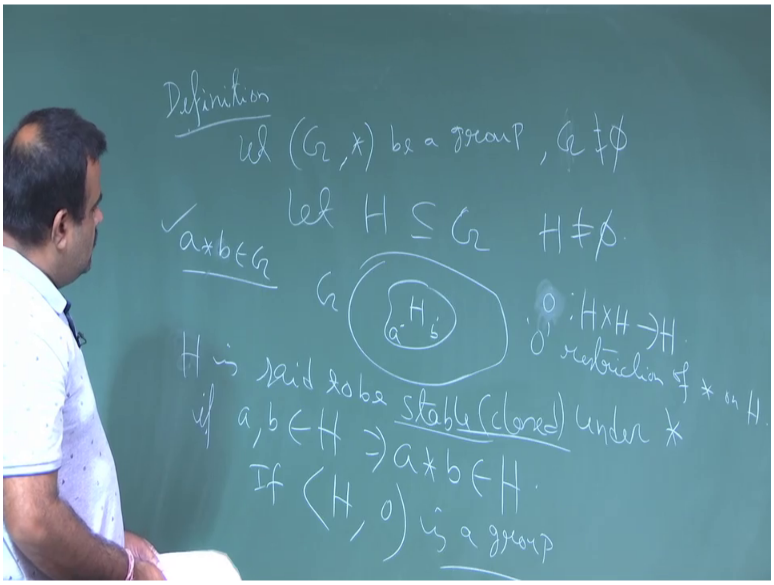

So, this is the definition now let G be a group, which is basically G is non empty and let H be a nonempty subset of G, where H is also nonempty, ok. So, this is our G group G set this is our H set. So, H is a subset of G.

So, now, we define the. So, G is having this operator now we define the induce operator of induce operator of this star on H. So, how to define? Now this star is said to be stable star is said to be this is G star is said to be stable or in other word it is called closed also, close under sorry H is. H is said to be stable or closed under star if and only if you take any two elements a b form H this is also an element in G and we know if this if you take two elements from a, b by H a, b now these are also element of G.

So, a star b will be element of G this we know because G is a group. So, the closure property is satisfied this is, but it need not be an H again, but we want this to be in H. So, that is the stability that is called stable or closed under this if implies a star b is also in H. So that means, H is closed under this star, ok. We know this will happen because this is a group G is a group. So, if we take any two element a, b then a star b will be belongs to G because of closure property, but we want to be the this property is in H also, ok.

So, then we can say this H is stable or closed under star and if we if this is closed under star then we restrict our operated star on H only. If we just think this we are operating the star on H only then we can say the star we can have a another operator this is a some different operator. This is basically what? This is a restriction on restriction of star on H, ok. This is basically the operator star, but the elements are coming from H this is a binary operation element are coming from H. So, this is the induce operator on this, in induced composition on this, anyway this also we can write f star inside of this.

Now, if this is form a group this along with this induce operator if this is a group then it is called a subgroup because this is a subset, then it is called a subgroup of G. So, let us formally define this.

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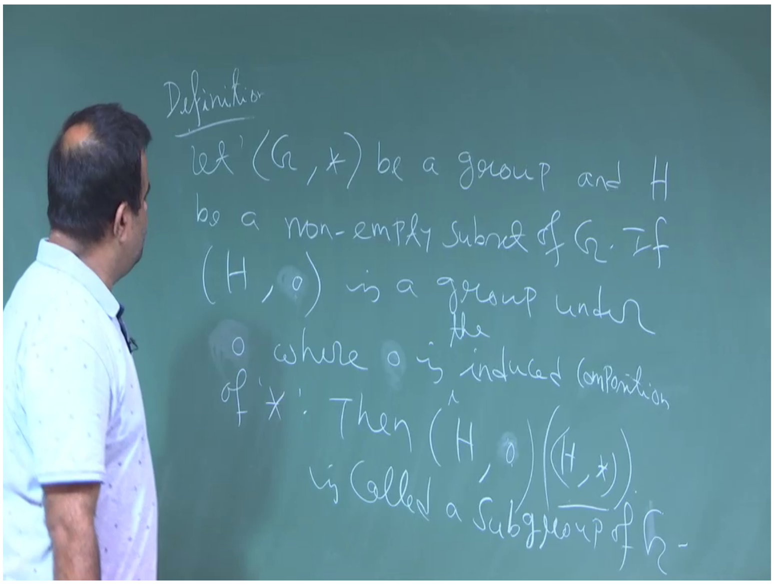

So, let this is the definition of sub group let G be a group nonempty group and H be a be a nonempty subset of G subset of G. Now, if H is also form a group is a group under the under star, where the star is the induce operator on H only, star is induce or we can say this is different operator but, it is basically the same operator all the thing we are restricting our domain to be H only induce where this is where this is the induced operator or induced composition, this binary composition induce composition of star, ok.

Then H star we can say star because when we say star you have to be little careful here because star is usually the binary composites on G, but here we are restricting our self on H ok. So, that is why better to use something other symbol other this small o kind of symbol to indicate that this is a operator which is induce operator of star on H, but anyway we can just this or sometimes we can just write star this, this is also fine, only thing when we write this we have to be careful that this is our domain is in from H only. Then this is called a subgroup of G because this is subset and the subset is also form a group under that induce operator. So, then it is called a subgroup of G. So, this is the definition of sub group.

Now, we want to have some example on the sub group, ok.

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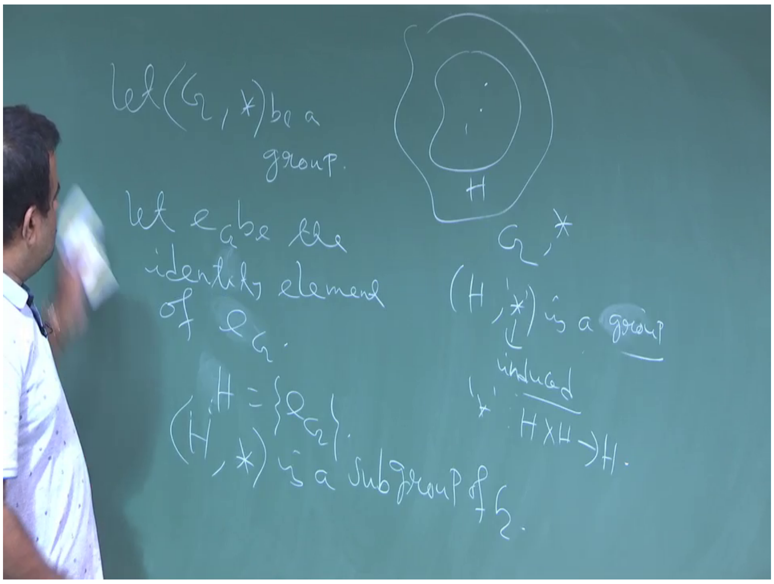

So, the obvious subgroup is the singleton set e. So, this is basically our G now we say and we have a star upper a composition on G, now this is a subset on this now we talk about the induced subset. So, if this is also a if this is a sub group, this is a group and then we call this a subgroup of G and here this star is induced because for this star we will take two element from this. So, this star is basically H, the domain is change in basically it is a composition on this is a binary competition on H, ok.

So, now let so, let G be a group now let e is the be the identity element, identity element of G. Now, we write as this way because now we have another set e G identity element G. Then the singleton set this is if we consider H to be the singleton set e G, then this will form a group then this is a sub group, this along with this star is a subgroup of G. We can easily check that because closure property is satisfying if you take this start this it is itself associativity is also ok. Identity is already there this is the identity, they may inverse the identity element has his own inverse I mean it is inverse itself. So, this is a subgroup. So, this is the singleton subgroup now, we take about take some example of other subgroups.

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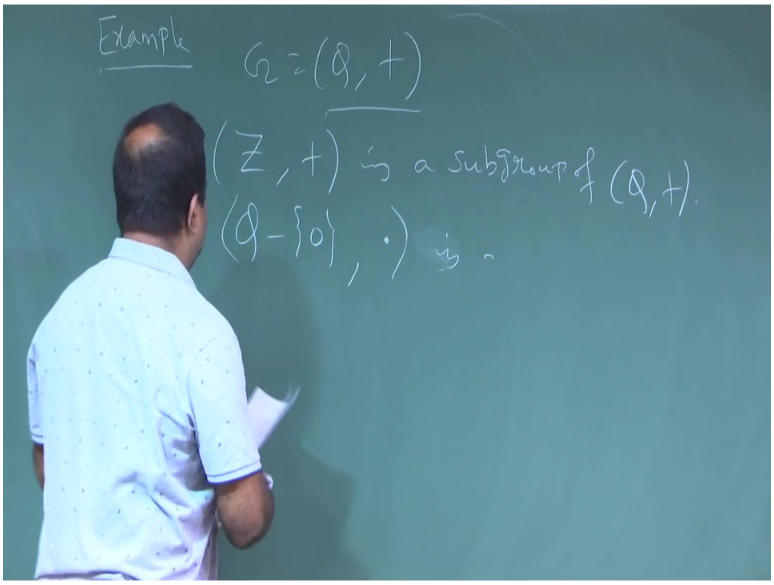

Now if we consider say. This is examples of subgroups say we consider the group G to be say Q plus our set is set of all rational number. Now if you take subset Z from Q this is a subgroup of Q plus this is one example. Another example is say if we take just Q, if we omit this then this will form a group.

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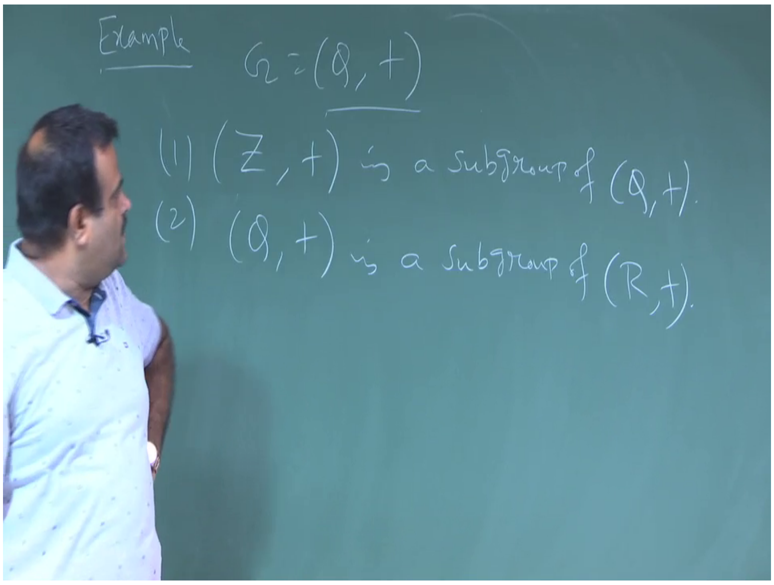

Or if we take just this Q if a subgroup of R set of all real number this Q is the set of all rational number. Now, we talk about there are many other examples also you can take the like yeah we can take some example like symmetric group.

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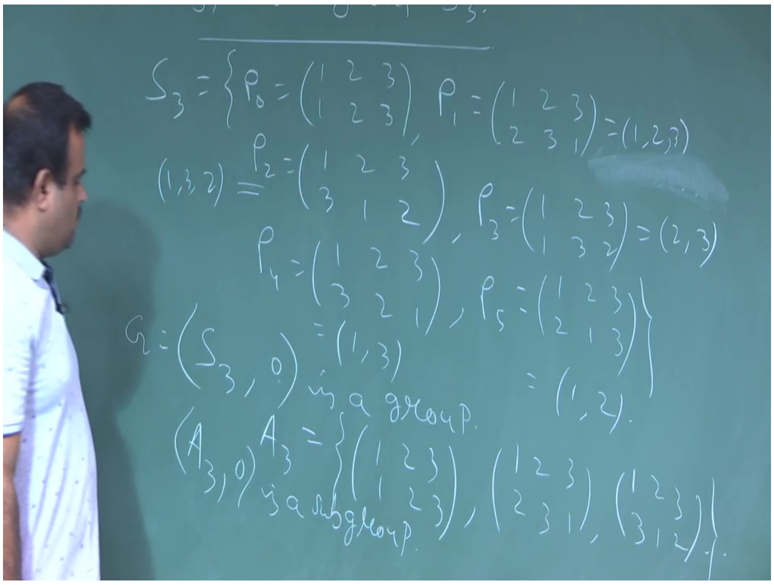

Say symmetric group of S 3. S 3 means this is the set of all permutation of order 3. So, how to define this? So, S 3 consists of this permutation. So, this is 1 2 3 this is the identity permutation then this is the 1 2 3 this is going to 2 3 1 and then 1 2 all possible.

So, how so, how many permutations are possible? So, there are 3 elements of factorial 3 6 permutations are possible.

So, let us just write all possible permutations 3 1 2 this is denoted by P 2, P 3 is basically 1 2 3. So, this is P 1, 1 2 3 then 1 3 2 and then P 4 1 2 3 it is basically 3 2 1 and then P 5. So, it is starting from 0 so, there are six permutations. 3 2 1 and this is basically 1 2 3, this is basically 2 1 3. So, these are the set this is the set this set this denoted by S 3. So, this is the set of all possible permutation on 3 element 1 2 3, ok.

Now, if we write this as a cycle, we know that every permutation can be written as a product of transposition like cycle of order 2 first let us try to write this as a cycle. So, this is basically a how to write this as a cycle. So, 1 is going to 2 and 2 is to is going to 3 and again 3 is going to also so, this is a cycle of order 3 now here. So, this is basically what? This is again 1 is going to 3 and 3 is going to 2 and 2 is going to 1 and here this is basically a 2 is going to 3 and 3 is going to 2 and here it is basically 1 is going to 3 and 3 is going to 2 and here it is 1 is going to 2 and 2 is going to 1. So, these are the cycles. So, this is the cycles of length.

Now, we know this is that this is a group under the composition of two these are all bijective mapping and these star is the composition of two bijective mapping. So, if you take two permutation if you composite it will give us a another permutation of same order. So, it is also a composition of two bijective mapping bijective mapping. So, we know this is a group this is an group. Now, we want to have a subgroup of G of this group. So, this is our G group now if you take all the even permutations like these are the even permutation because this will be again written as product of two sorry all permutation this 3. So, now if you take A 3 so, this is basically no sorry if you take all the even permutation, this is the authorization. Odd permutation means if it is written as product of this.

Now, even permutation is so, this is this can be written as 1 3 and then 3 1 like this. So, every now this is basically this is basically a even these are all these three are all odd permutation and this these three are even permutation. So, what are the even permutation like identity permutations 1 2 3, 1 2 3 and then this one this P 1 and then another one is basically this one P 2. 1 2 3, 3 1 2 this is P 2 3 1 2. So, this will be a subgroup of this, ok.

This is a because this we can. So, again this is a group of G. So, this is called subgroup of this is a subgroup of this S 3 this is another example of subgroup.

Now, I will talk about some of the properties of this. So, we will come back to this again.

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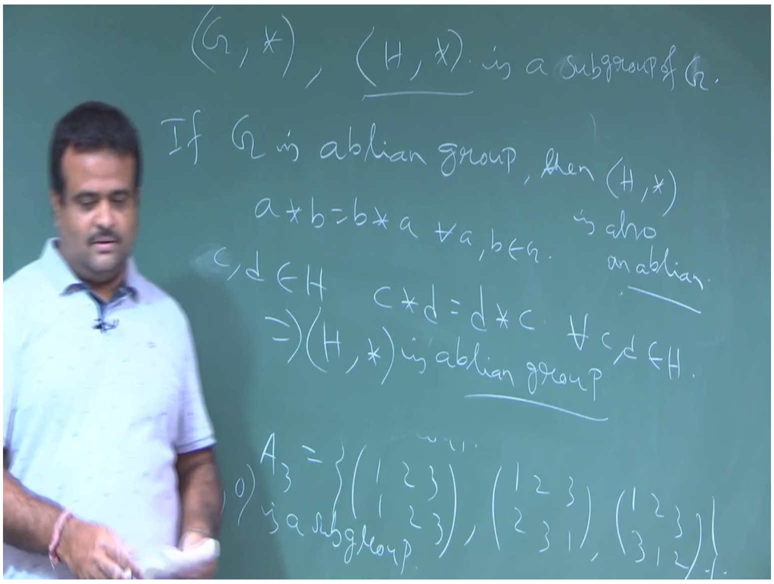

Now, first property is suppose G; G is a group and H is a subgroup of G is a nonempty subgroup of G. Now, it is the theorem is telling if G is abelian or commutative G is abelian group then this will be also an abelian also an abelian group. Why? because if G is abelian means what a star b is equal to b star a for all a, b now, how to show G H is abelian if you take any two element say c, d sorry c, d form H then we know c star d they are also element in G. So, c star d will be d star c and this is true for all c, d belongs to H. So, this implies H is also abelian group.

But, the converse is not true; that means, if we may have a subgroup which is abelian, but the original group is not abelian for example, this A 3 is a; A 3 we can be easily checked is A 3 is a abelian group, this is a subgroup of S 3, but S 3 itself is not an abelian group S 3 is not commutative because in general the bijective mapping is it sorry bijective composition is not a commutative operation. So, this is an example here we have a subgroup which is abelian, but the group is not abelian.

So, now we talk about some other properties like if they have a same inverse they may not have two different inverse or identity element.

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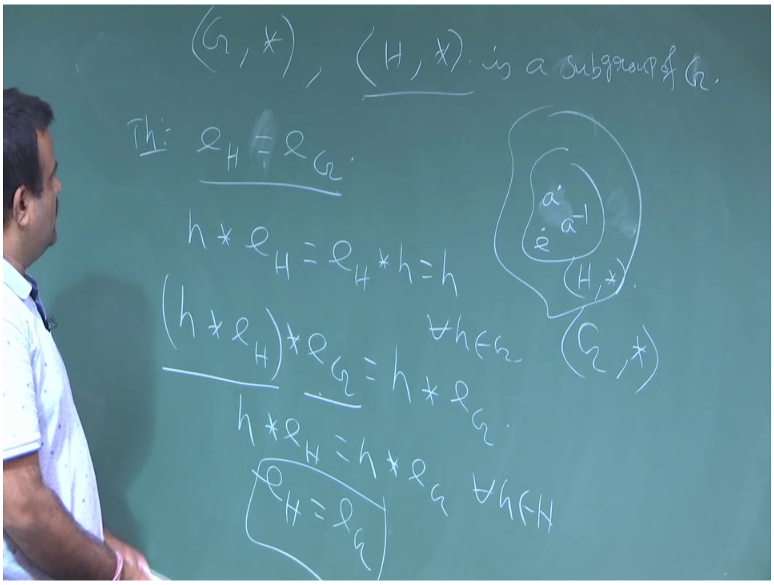

So, we have this group G and you have the subset H which is also subgroup under the same operation that induce operation. So, this is also a subgroup. Now, our claim is this identity element of G must be the identity element of H. So, they have same identity element. So, that this is another theorem. So, idea of identity element of G must be identity element of H. So, this is the theorem. So, they may not we should not have two different identity. In other word the id identity element of G is belongs to the H.

So, how to prove this? So, to prove this we can just suppose. So, we have to prove this. So, suppose they are not same they are different. So, we have a identity element here and we have a different identity element, which is not in a H set or maybe in H, but they are different. So, we have two different identity element.

Now, if we have this then what we can say? We know this e G is the identity element of this then we can simply say H star e H is basically equal to H star H which is basically H and this is true for all H. Now, if we apply both side by e G, which is basically H star e G now this is an element in G and this is the identity element of G. So, if we apply any element with the identity element this will give us the that element. So, H star e H is same as H star e G and this is true for all H. Now, by cancelation property we can just say H equal to e G. So, this is the this is how you can means that. So, they have the same identity element. So, the id so, identity element will belongs to the H only they are the same they say are the same identity element, which is we know the identity element is unique in the group, ok.

Now the inverse say inverse is also now inverse we cannot say because see we if we take an element over here then a inverse has to be belongs to H and you know that is the image element, ok. Now, we now we talk about some necessary and sufficient condition to be a subset nonempty subset to be a subgroup of G ok.

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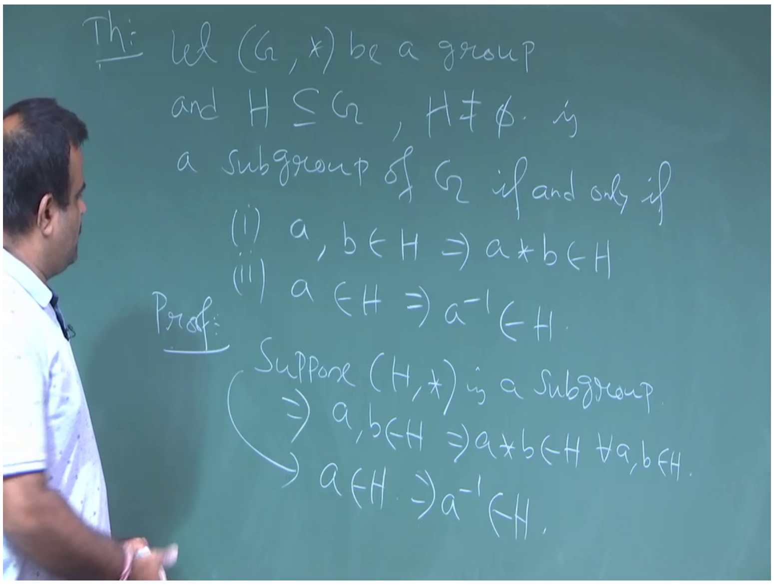

So, this we write in a theorem form or lemma form. So, this is telling let G be a group and H is a nonempty subset so, that means, H is not equal to 5 and H is called H is a subgroup of G if and only if this two property satisfied. So, a b belongs to H this is the closure property and the second property is a belongs to H imply a inverse belongs to H. So, this is the, this is a theorem where ok.

So, now, how to prove this? There is two part of this theorem necessary part and sufficient part if and only if. Now, first we assume that this is a subgroup. Now, suppose this is the first part let us suppose H is a subgroup, then we show this two satisfy, then this implies if we take a, b belongs to H this implies a star b belongs to H for all a, b because of closure property.

Now, also since this is a subgroup, then if a belongs to H then a must have an inverse because it is a group basically it is a group and it is a subset that is why it is called subgroup. So, this implies a inverse belongs to H. So, every element has a inverse in a group. So, H is a group. So, every element has a inverse. So, these two properties satisfied.

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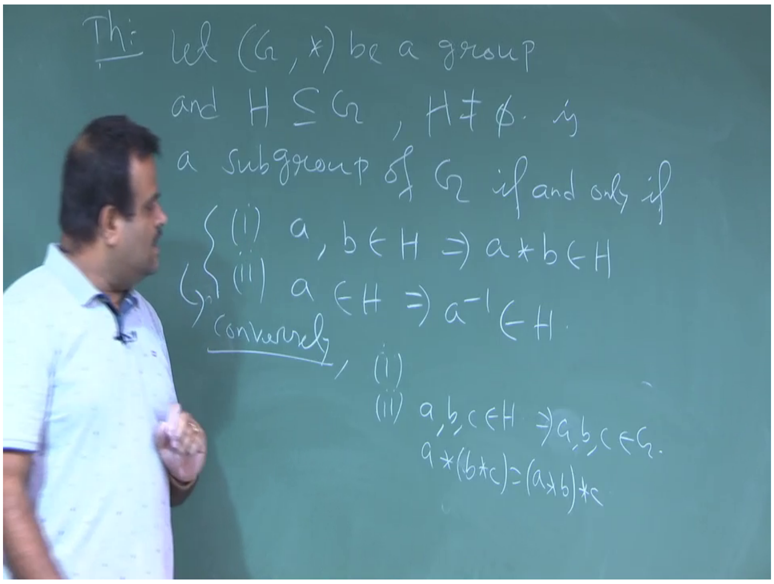

Now, the other way converse. Converse means suppose H is a subset non empty subset and this true is true converse. Suppose these two is satisfying then we have to show that H is a subgroup. So, now first property is telling us the closure property, if we take any two element then they are they are a star b belongs to H. So, closure property is satisfying. Now this is closure is done associativity; associativity is also because if you take any three element form H, they are also from this implies because H is a subset of G then we can write since G is a group, we can write this because G is a group. So, in the group the associativity satisfy G is a group. So, star is associate in G. So, they if a, b, c are in H. So, they will be also in G so associativity is straightforward.

Now, the identity element we need to check the identity element, whether this belongs to H or not.

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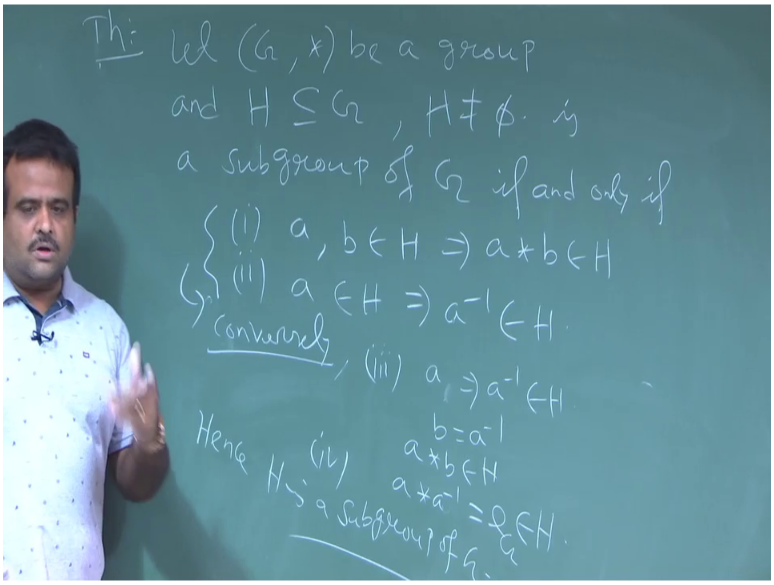

So, existence of identity, ok. Now, if we take we know the if you take an element a, now just this if this implies a inverse also belongs to H. Now, if you take a inverse b to be a inverse now we know this a b belongs to H means a star b belongs to H. So, this is nothing, but a star a inverse this is nothing, but e. So, e belongs to H. So, the identity element of the group is also an identity element of the, of identity element belongs to H. So, e is the so, identity element of that group is same as identity of this subgroup also. So, this is not a subgroup all these three property satisfied, now we need to check the inverse.

Inverse is also because from this result because if a is belongs to H that a inverse is also belongs to H. So, from this result we can say inverses also exist. So, every element has the inverse. So, hence this is a group, H is the subgroup of G, ok.

So, we will talk about more on the subgroups in the next class.

Thank you.