Constrained and Unconstrained Optimization

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Lecture – 15
Sensitivity Analysis- I

In this class, we are going to study the post optimality analysis. Or sometimes we call it as the sensitivity analysis also. If you see your post optimality analysis is the study of changes in optimal solution due to changes In LPP parameters.

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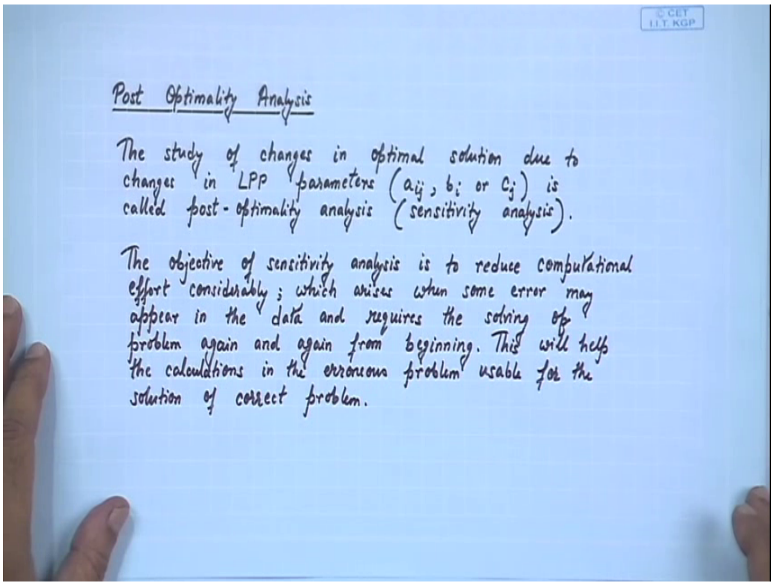

The parameters mostly are a i b i a i j b i or c j, which is called post optimality analysis or sensitivity analysis. So, basically we want to see the effect of the solution optimal solution, whenever we are making changes in different parameters. Mostly the parameter in the objective function that is c i j, the matrix that is a i j that is coefficient matrix and also the resource that is b j.

So, whenever we change these parameters actually what happens, or what is the change in the optimal solution, and you may ask why this is necessary. The objective of the sensitivity analysis is to reduce computational effort a computational effort considerably, which wishes which arise when some error may appear to the data to the data and requires the solving of the problem again and again from the beginning. This will help the this will help the conclusions calculations in the anonymous problem usable for the solution of the correct problem.

So, basically if you see sometimes what happens you have a problem, but you have to change the parameters and you have to run the parameters. And whenever we are running the parameters it is computationally it takes lot of time. So, if we can set the time by predicting something that if this variable value is changing from a particular range, say from minus x 2 plus x then the optimal solution will remain unchanged. In that case you do not have to compute the problem again. So, you are saving basically the computational time and for this reason sensitivity analysis is very much required for us.

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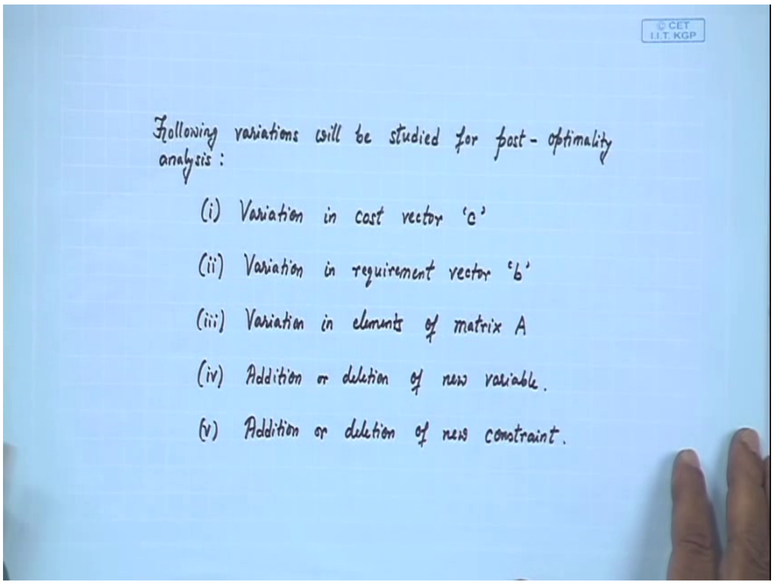

What we are going to study here is that, following variations will be studied in post optimality analysis, number one is variation in the cost vector c, number 2 is the variation in the requirement vector b, number 3 variation in the elements of matrix A. Then these are the 3 parameters which have been used if you remember our LPP problem is in the standard form maximize z equals cx subject to A x equals b.

So, the parameters involved are c b and A. So, if I vary these parameters; what is the effect? Number 4 is addition or deletion of a new variable. Means I can add a new variable or I can delete a variable what will be the effect and addition of a new constraint or deletion of a new constraint. So, we will study these things in our in this particular course.

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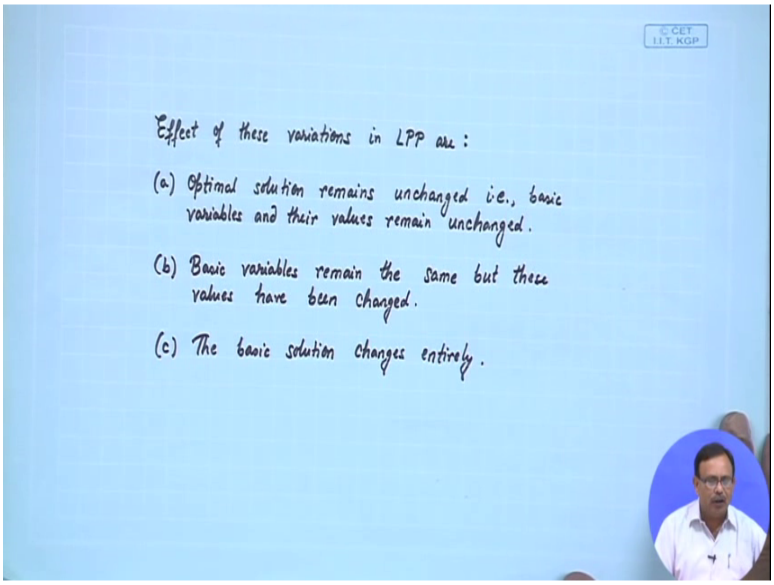

What is the effect of these variations? The effects we say that the optimal solution remains unchanged, that is your optimal solution will not change at all. The basic variables and their value will remain unchanged and if we vary second point is basic variables remain the same, but their value may be changed that is in the final basis whatever variables you got after variation of the parameter the basis variable will remain same, but their value may is change and number third point is the basic solution may change entirely that is you may get a new solution at together.

So, these are the variations which you may get in your computation. That is number one is the; your optimal solution may remain unchanged, and where as the basic variables and their values also will remain unchanged. Your basic variable values may change although the basis will remain, as it is and the entire solution may change.

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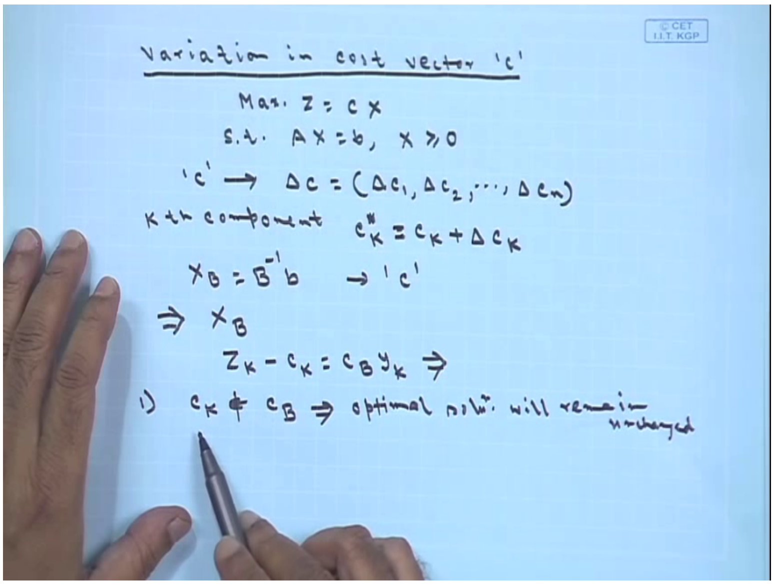

So, let us see the first one. That is variation in the cost vector c, variation in cost vector c. And as you know your cost vector is associated with the objective function that is if I write down in standard from the LPP maximize z equals cx subject to A x equals b and x greater than equals 0, that non negativity condition.

Now suppose the parameter c is change to (Refer Time: 05:58) by is changed by delta c, delta c equals that is the change in thus vector c we are telling that there will be a change of delta c. Which you can write down delta c 1 delta c 2 like this way delta c n. So, the kth component you consider just let us take only one component. So, if I take kth component of the modified cost vector I can write down kth component of the modified cost vector C k star I am denoting it, old value was C k new value is C k star plus del ck.

So, your old value C k has been change to C k star where the new value C k star is equals to C k plus delta ck. Now what is the basic solution? Basic solution of your problem is X B equals B inverse b which is fixed. So, if you see this solution, B inverse b which is independent of the cost vector c which is independent of the cost vector c. So, from here this implies that the X B will remain unchanged. X B will remain unchanged that is current solution whatever is there that will remain the basic solution, is it clear. I have changed the c vector the kth component of c vector by C k plus delta ck, but we know the basic solution is X B equals B inverse b which is independent of the vector c therefore, X B will remain unchanged. That is the old solution will be the solution after the change from C k to C k plus delta ck.

So, if I make any variation in the cost vector or in the cost vector of the objective function c, then the basic solution will also remain unchanged. And also you know it Z k minus C k this is equals C B into Y k. If you remember Z j minus C j earlier we have computed are C B into Y B. Y B is nothing but the base variables x one x 2 x 3 which I am denoting here as y k. So, since this Z k minus C k value is dependent on C B. Therefore, optimality criteria may change please note that optimality criteria may change may not change. So, 2 conditions may arise one is if your C k does not belongs to C B. This is one condition C k may not be belongs to C B at all in that case optimal solution will remain unchanged optimal solution will remain unchanged.

So, if C k does not belongs to C b. So, it is not affecting the values. So, optimal solution will remain unchanged.

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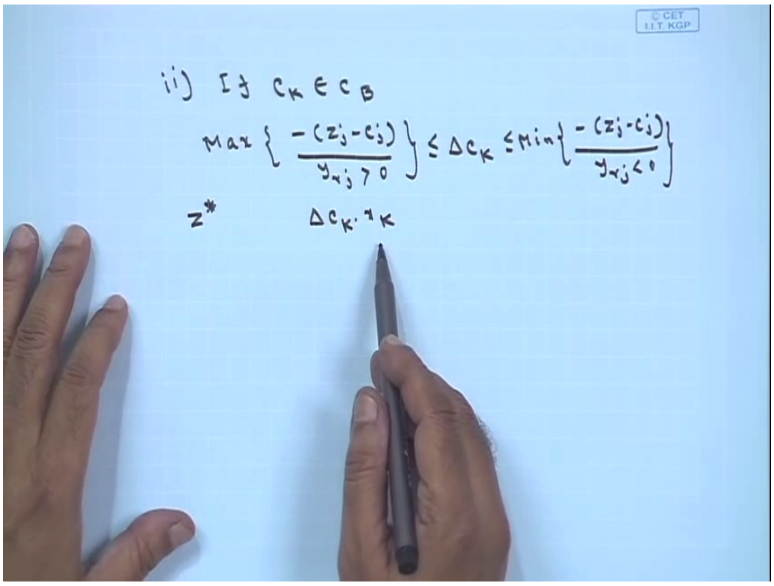

And number 2 is if your number 2 is if C k belongs to C B, and if this is true maximization of minus of Z j minus C j divided by Y rj which is greater than 0 less than equals delta C k less than equals minimum of minus delta j minus C j divided by Y rj. Where i Y rj may be less than 0. Then if your delta C k value lies in this particular range, maximum of this and minimum of this one in that case the optimal cost z star may will be improved. And so, optimal cost will be improved. And how much it will improve? It will be improved as delta C k into xk, because your cost has increased your by delta ck; that means, you will get more cost or more profit and what is the extra amount extra amount is delta C k into xk.

So, let us begin start from the beginning whenever we have a change in the cost parameter that is c. In that case there will be since from here, X B equals B inverse b which is independent of c. So, there will be no change in the optimal solution. Although Z k minus C k since this is equals C B into Y k. This is equals to C B into y k. So, in that case optimal solution may change when whereas, because here 2 cases may arise C k does not belongs to C B, in that case optimal solution will remain unchanged. Whereas, if C k belongs to C B in that case C k belongs to C B and if this is true delta C k lies between maximum of this value and minimum of this then the optimal cost z star will be improved by the quantity delta C k into xk.

So, basically the feasible solution will not change at all whenever you are changing the value of the cost parameter c, but if you change the cost parameter c then it may happen that your optimal cost may improve and what is the improve quantity improve quantity will be delta C k into xk.

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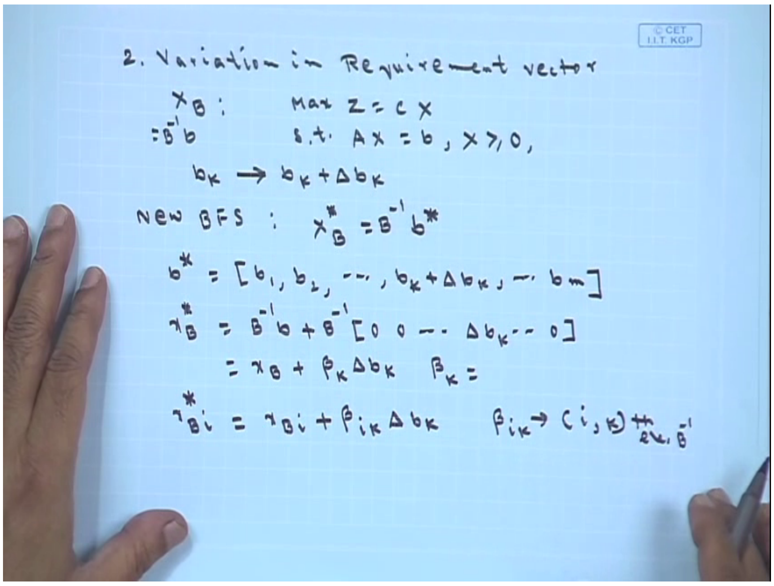

Now, let us go to the next one that is number 2 is variation in requirement vector. So, whenever you are talking about variation in requirement vector. So, suppose your X B is the optimal basic feasible solution of the problem, maximize z equals cx subject to A x equals b x greater than equals 0 and as usual a is the m cross n matrix.

Now, we have told requirement vector requirement vector means this b. So, suppose your bk is changed to bk plus delta k. Suppose your delta bk bk the requirement vector b the kth component of the requirement vector b changed from bk to bk plus del bk. So, what will be the new basic feasible solution? New basic feasible solution will be say X B star equals B inverse b star this is equals to B inverse b star where your b star is nothing but this one, b 1 b 2 like this way and your bk has been change to bk plus delta bk and b m. So, what is your X B star? Your X B star will be B inverse b your X B star is B inverse b star if I write down this thing. Then I can break it B inverse b plus B inverse all this will be going out 0 0 like this way delta bk and this will be 0.

So, this is nothing but this B inverse b is nothing but your X B, B inverse b because your X B is equals to B inverse b originally. And then whenever I changed the kth component of bk, from bk to bk plus delta bk. The new basic feasible solution has become b is inverse in to b star. So, this is I can write down X B plus beta k into dell bk. Where beta k is equals to kth column vector of B inverse. Where beta k is equals to kth column vector of this B inverse into bk. So, please note this beta k is the kth column vector of B inverse. And we can write it in the other way also. So, therefore, I can write down X B i star this is equals to X B i now I am generalizing it. So, beta ik into delta bk. Where your beta ik will be the I comma kth element I comma kth element of B inverse where beta ik is the I comma kth element of beta inverse.

Now, let us check the feasibility of this new solution. Feasibility means your x bi star should be greater than equals 0 for all i equals 1 2 m.

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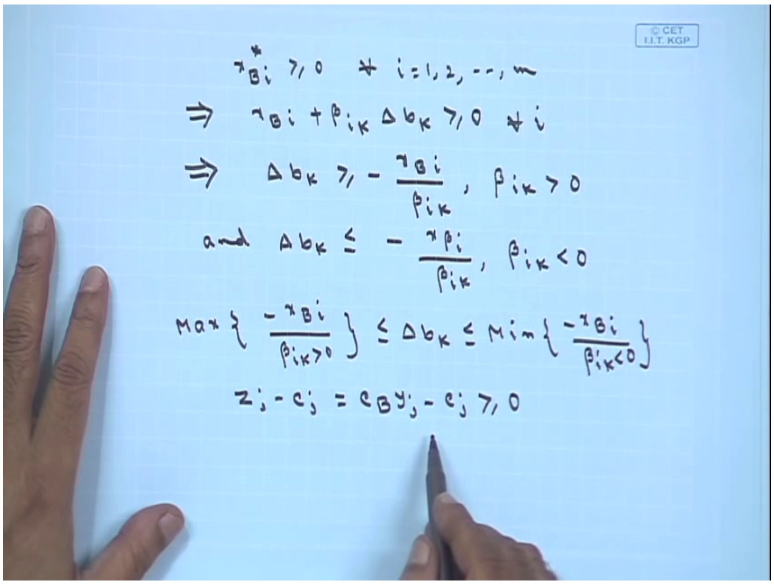

So, xbi star is greater than equals 0, which implies xbi from the value by substituting the value of xbi star from the earlier one this value. So, it becomes xbi plus beta ik into delta bk which is greater than equals 0 for all i equals 1 to m which implies again I can write down delta bk is greater than equals minus xbi by beta ik. Whenever beta ik is greater than 0 and your delta bk will be less than equals minus x beta i by beta ik whenever your beta ik is less than 0. So, from this expression I can write down these 2, one for beta ik greater than 0 or beta ik less than 0.

So, if I combine these 2 the condition for feasibility will be maximum of minus xbi by beta ik which should be greater than 0 is less than equals delta bk, which is less than equals minimum of minus xbi by beta ik, where beta ik is less than 0. Now it is very easy to see one thing that the requirement vector the new solution will remain optimal. Because the optimality condition is independent of beta your optimality condition is this one Z j minus C j this is equals C B yj minus C j, which should be greater than equals 0. For optimality this condition should be satisfied and this is independent of b therefore, the we can say that any variation in the requirement vector will not affect the optimality criteria. Will not affect the optimality criteria, but it may affect the feasibility criteria. Please note this one. If your delta bk lies in this range in that case the old solution will remain feasible and optimality criteria will not be changed.

So, please note that for the requirement vector b if the variation in b lies in this range in that case your b value of b in that case the old optimal solution will remain unchanged, but otherwise if we does not lie then the feasibility condition may change. So, please note that your optimality condition will never change will remain unchanged, but feasibility condition may change if your value of the vector b does not lie in that range that is maximum of that and minimum of that. So, if I know beforehand you see the advantage of this testing the sensitivity because if I have made a small change and if I see that it is lying in my range. I do not have to re-compute the entire problem from the beginning from this theory itself we can directly tell that the solution is remains unchanged or solution will be the same. So, that is the basic criteria for this one.

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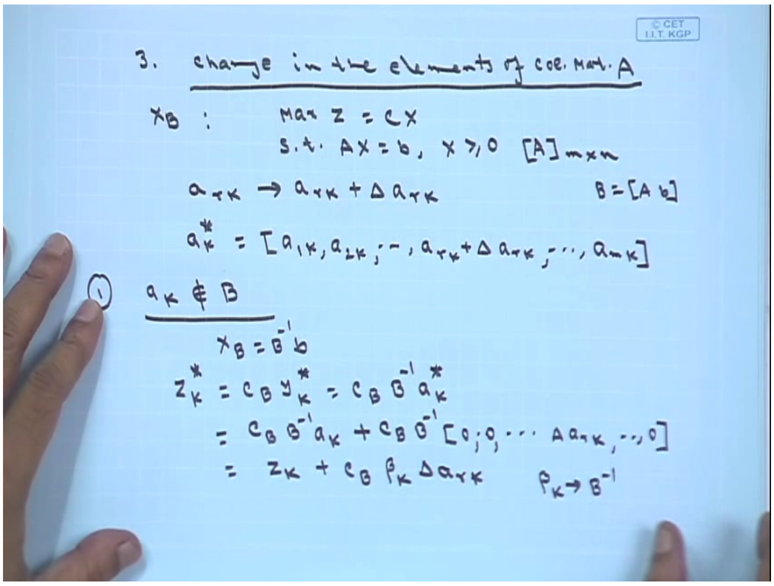

Now, let us come to the third point. That point is change in the elements of change in the elements of coefficient matrix A, other 2 first one we have done change in the vector c. Then we have done the change in the vector requirement vector b now we as we want to say the effect of making changes or variation in the coefficient vector a. So, again your we are assuming that X B is the optimal basic feasible solution of the problem, maximize z equals cx subject to A x equals b x is greater than equals 0. Where A is basically m cross n matrix. So, initially X B is the optimal solution of this problem.

Now suppose one element of a see ark is change to ark plus delta ark ark is change to ark plus delta ark. Then the modified vector ak star can be written as this one a 1 k a 2 k like this way, ark plus delta ark like this way amk. This is the modified vector. Again 2 cases may arise that is your ak does not belongs to b or ak belongs to capital B your capital B is nothing but this matrix may be your capital B is nothing but this one. So, case one where ak does not belongs to capital B. If ak does not belongs to capital B and your X B equals B inverse b. Then optimal values of the variables will remain unchanged. Because it is B inverse b and your ak does not belongs to this b b so; that means, it is not affecting therefore, the optimal values of the variables will remain unchanged.

Now let us see Z k star what is Z k star Z k star is C B into Y k star and this is equal C B into Y B is nothing but B inverse into this one B inverse into ak star. So, this equals I can write down C B B inverse into ak star is this one. So, from here I can using earlier deduction I can write it C B B inverse ak plus C B B inverse into just like a dr 1 0 0 like this way delta ark like this way 0. And c B B inverse ak this is nothing but the kth value of the z. So, this is Z k plus C B into beta k into delta ark. Where again beta k is the kth column vector of B inverse. So, beta k is the kth column vector of B inverse after multiplication.

So, this we are writing that your Z k star is after changed it is becoming Z k plus C B into beta k into delta ark. Now for optimality your Z k minus C k should be greater than equals 0 we know it for optimality Z k minus C k is greater than equals 0.

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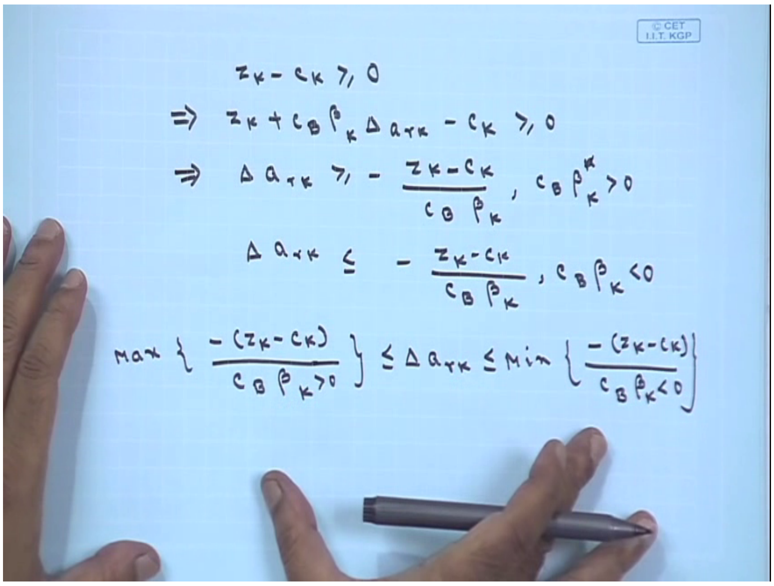

So, this implies that Z k plus C B into beta k beta k, delta ark minus C k this is greater than equals 0. Or from here I can say delta ark is greater than equals minus Z k minus C k by C B into beta k. Where C B into beta k is sorry beta k is greater than 0. And also delta ark will be less than equals minus Z k minus C k by C B into beta k for C B into beta k less than 0. And delta ark will remain unrestricted if the C B into beta k equals 0.

So, combining we can write it again in this form maximize of minus Z k minus C k by C B into beta k. Where the C B beta k is greater than 0, is less than equals delta ark less than equals minimum of minus of Z k minus C k by C B into beta k which is less than 0, which I am writing like this way. So, therefore, what is the range of this one. The range is if delta rk lies in between maximum value of these 2 minimum value of this. In that case what we are finding out that even if for variation in delta rk.

There will be no change in the optimal solution. And please note that this we are doing it only whenever your ak does not belongs to b. So, if ak does not belongs to b and if the variation in ak lies in this range in that case optimal solution will remain unchanged otherwise we will there may have some changes. So, whenever C B belongs to capital B that we will see in the next class.