Model 4

Lecture – 25

Criteria for Divergent Sequence

Course

On

Introductory Course in Real Analysis

So this is in continuation of my previous talk, where we have discussed infinite series of real numbers and its convergence or divergence. The discussion was supported by giving few examples. We will continue the discussion further by giving few more convergence criterion of for infinite series of positive real numbers.

First we proved an important theorem up on convergence criteria of the series Sigma except 1 to infinite and then will drive as a corollary many other results which are helpful in identifying the nature of the series. So these are now there are some criteria for the divergence.

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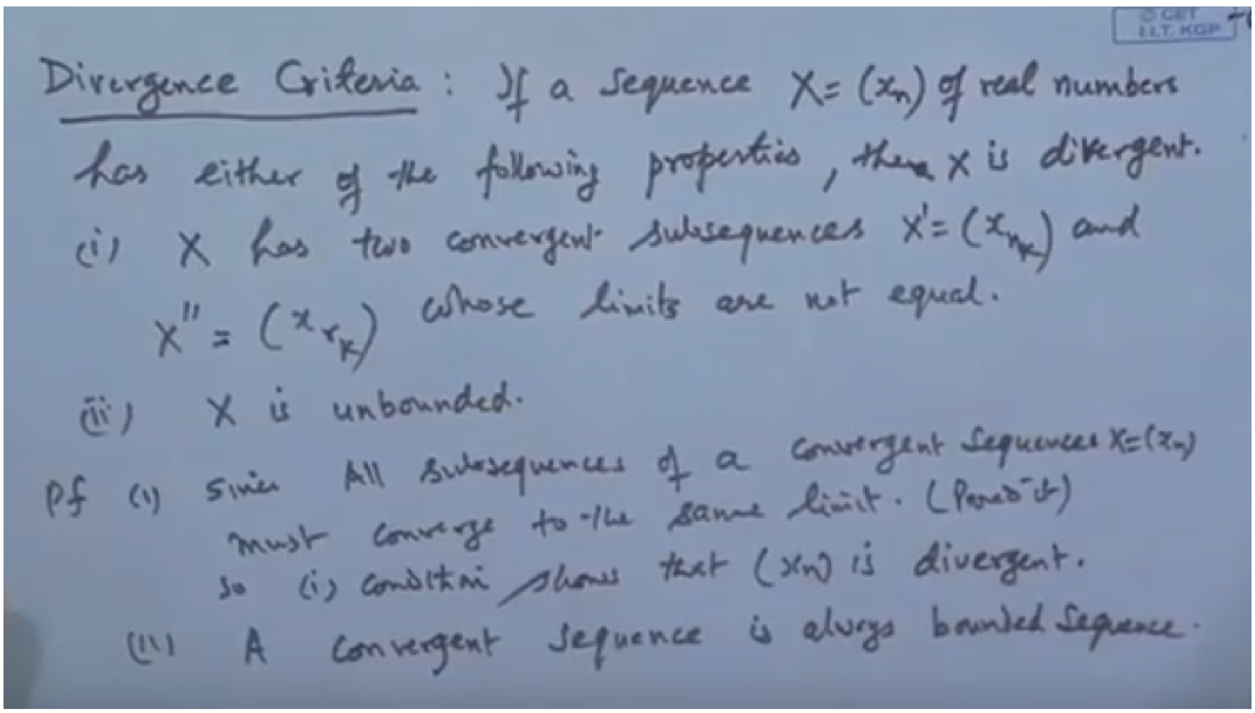

so this we call it as a divergence criteria, divergence criteria if a sequence, sequence X, say xn of real numbers, of real numbers has either of the following, of the following either of the following properties, then X is divergent, divergent. What is the first is? X has two convergent, two convergent subsequences, two convergent Sub sequences X dash, which is say X n K and X double dash, say X R K, whose limits are not equal. And second one is X is unbounded, okay. the proof is very simple, if a sequence xn of real number has either of the following property then X is diverging, so what is this property first property says if the sequence X has two convergent Sub sequences, whose limits are not equal then the sequence will be considered as a diverging sequence or second one is if X is unbounded that is the limit of xn will go to either plus infinity or minus infinity, then also sequence is considered to be a diverging sequence. the proof follows immediately from the fact that if suppose X is a convergent sequence then in case of the convergent sequence and this subsequence of the convergent sequence must be convergent and converge to the same limit because the limit is unique, so if X any sequence have a different limit along the different sub sequences different paths, then the sequence cannot be a convergent one so it has to be diverging, similarly every convergent sequence is a bounded sequence, so if a sequence is unbounded it cannot be convergent so proof first is follows since, since all sub sequences of a convergent sequence, of convergent sequence X equal to xn, say all subsequence of convergence you can must converge to the same limit. This is the criteria for the convergent is it not? That what we have proved also this is proved earlier proved it, so if sequence has a subsequence if it does not converge then obviously it will be diverging. So first criteria first condition shows, first condition shows that the sequence xn is divergent, divergent.

Second follows since a convergent sequence is bounded is always bounded sequence, so if a sequence is unbounded then it must be it must not be a converge it will not be a converging sequence, now here we are not saying boundedness, because even a divergent sequence may be a bounded sequence, so we are not saying what we are saying is a convergent sequence will always be bounded sequence so once it is unbounded of definitely that sequence will be a diverging sequence. So that's proof for Module, okay. So these criteria I think we have given already examples taken.

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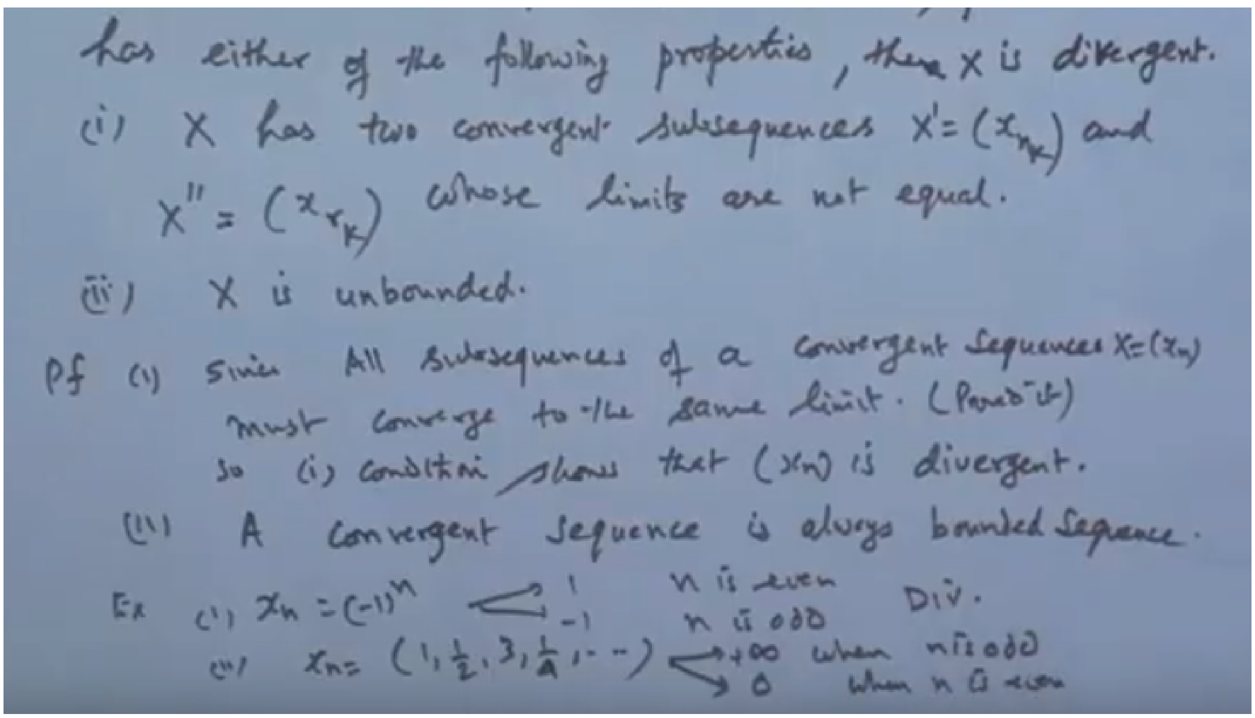

for example, if we take a sequence xn which is minus 1 to the power n this is sequence then this has a to limit 1 and minus 1, when n is even and when n is odd, so it is a diverging sequence and second if I take the sequence xn to be say like this 1,1/2, ,3, 1 by 4 and so on so this sequence when there is odd numbers it will go to infinity, when n is odd and when n is even, then it will go to 0 because 1/2 1 by 4, 1 by 8, 6**, 1** by 8 and so on so when n is so it has the two limits, okay. Therefore, it will not be a divergent sequence. now we have for the monotonic sequences we have a criteria that every bounded monotone sequence is convergent and that is called a sequence monotonic convergence bounded sequence of monotonic sequence always converge and based on this, we have a monotone sub sequence theorem that theorem says if xn is a real number then there is a sub sequence of n, that is monotone. in fact if a sequence is given then it's not necessary that sequence be a monotone sequence, but always we can identify at least a subsequence which has a monotone which will be a monotone so every sequence of the real number have a monotone subsequence,

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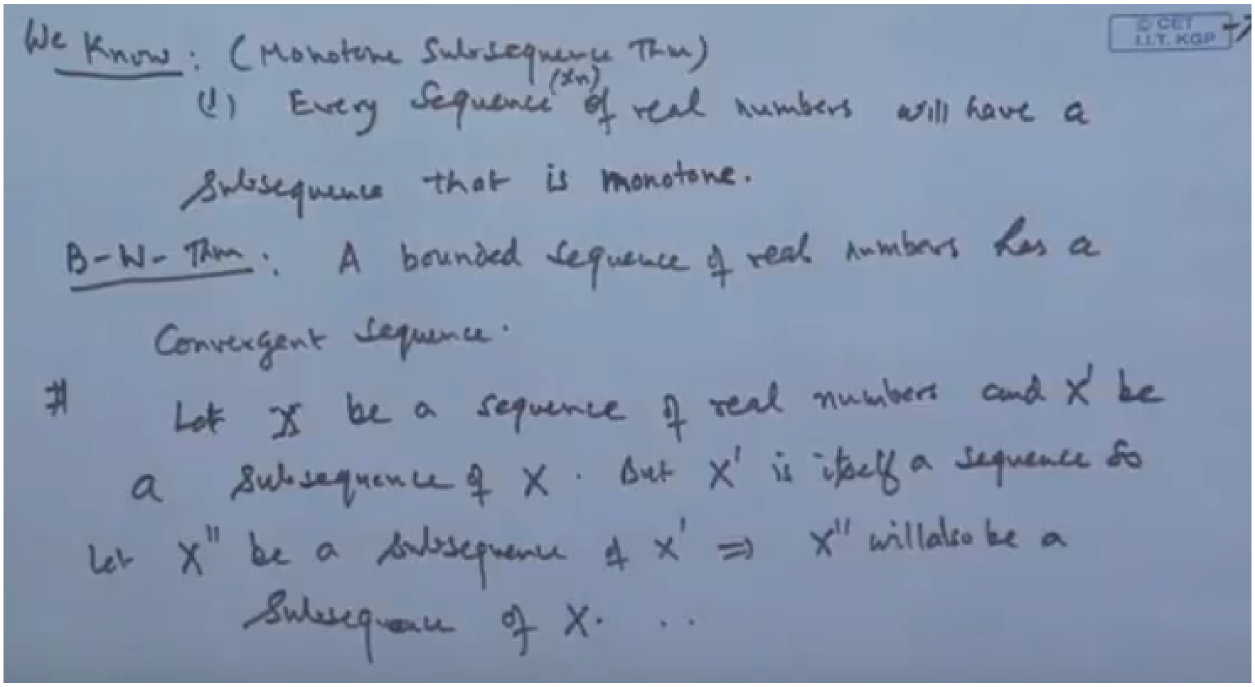

so that we know that every sequence, every sequence, every sequence of real number, of real numbers will have a sub sequence, will have a subsequence, that is monotone Either monotone increasing monotone decreasing and so on. so once it is monotone and if it is bounded then that particular sequence will be monotone sequence will be convergent but it does not mean that sequence xn will be convergent, because the sequence xn, does not have all these sub sequences which are monotone, so you cannot say the sequence itself is a convergent one, okay. only one particular subsequence which comes out to be a monotone sequence and if it is bounded sequence it will converge if it is unbounded it will diverge like that. so again the criteria for the monotone will not help you much, unless the sequence is monotone we are unable to identify where the sequence is converging or not and there is one result which also we have seen, theorem, this theorem we have shown a bounded sequence of real number, sequence of real numbers has a convergent subsequence, so both these reason that is monotonic sub sequence theorem, this is the monotone sub sequence theorem, and Bolzano west's theorem there gives a rough idea about this some of the sub sequences but it does not give that total idea about the entire sequence so we are unable to get this and whether the sequence is convergent, but the Cauchy has given this idea which is known as the Cauchy convergence criteria, which without calculating the limit of the sequence, which can tell whether sequence is convergent not. So for this we will develop first few results and then go. let x be a sequence of real number, real numbers and let x dash be a subsequence X dash is s and let X dash is this Sub sequence be a subsequence of X, now if we consider X dash as independent sequence then basically it is also a sequence, so again we can identify a sub sequence of X dash again so we can get let X double dash, but X dot dash, is itself a sequence so we can identify that so let X double dash be a sub sequence of X dash then obviously the element of x double dash are also element of X so obviously X double dash will also be, will also be a subsequence of X, is it okay? So this n continues like this till we get this, okay. Now we have a very interesting result, the result is,

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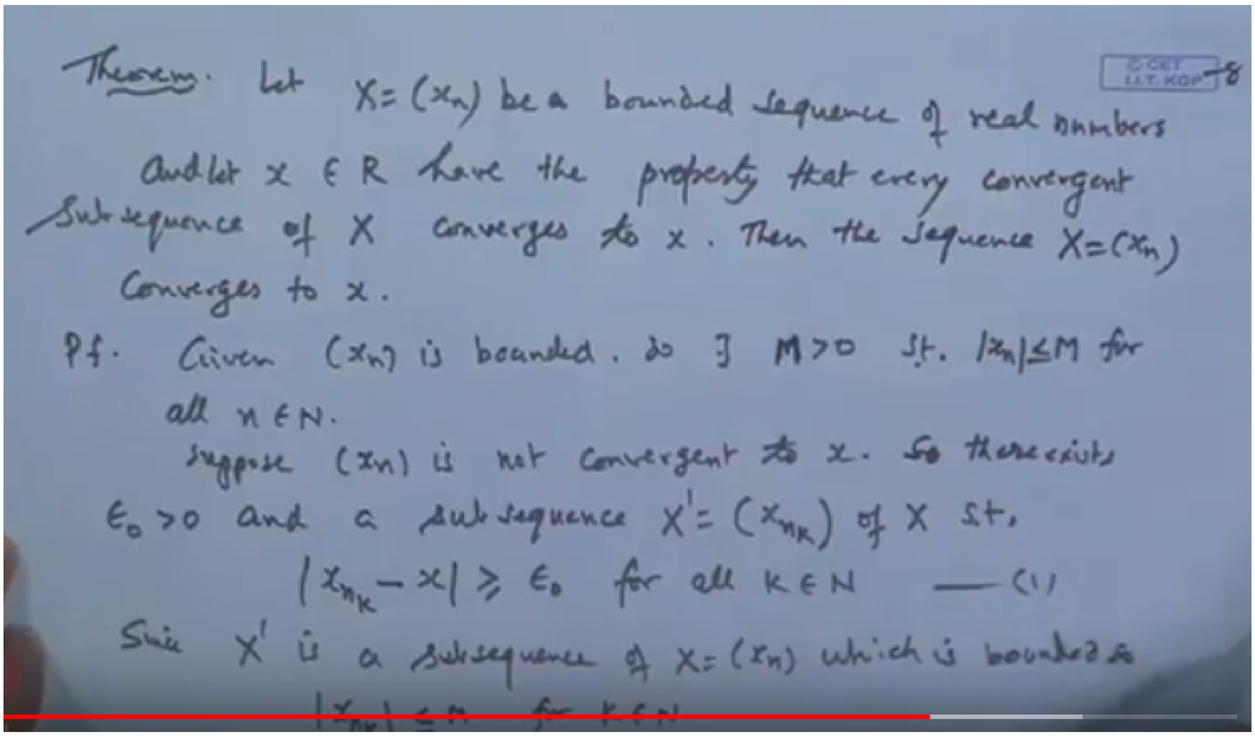

the theorem says, Let X which is xn, be a bounded sequence of real number, sequence of real numbers, bounded sequence of real numbers and Let X and X, is a belongs to R and let X belongs to R. have the property that every convergent subsequence, convergent sub sequence, every convergent subsequence of X converges because section converges to X and Let X belongs to have the property there every convergent sub sequence of X converges to X then the sequence X, then, the sequence capital X converges, which is xn, converges to X. Now this is the converse part of the previous results in the previously result we have shown that every if a sequence is convergent then all of its sub sequences will be convergent and converge to the same limit. Now this shows the converse part, suppose a sequence is bounded sequence and if all of its subsequences converge to the same point X, then the sequence must be a convergent one. So proof is now given that xn is a bounded sequence, given the sequence xn is bounded so it means all the terms of the sequence are dominated are less than equal to some numbers so there exists in M greater than 0, such that all the terms of the sequence is less than equal to M for all n, for all n belongs to N, because it is bounded. Now we wanted the xn is convergent so suppose this sequence xn is not convergent is not convergent to X, is not convergent to X is not a not convergent and to the X point convergent and converges to X and not convergent, ok. Then whatever once the sequence is not convergent does not converge to X so we will apply the criteria which we had discussed for the diverging sequence then by the criteria we can say so there exists so there exist, there exists an epsilon naught greater than 0 and subsequence and a subsequence X dash which is xn K of X such that such that X and K minus X is greater than equal to epsilon naught for all K belongs to a set of natural number n belongs to n is it so let it be 1, now X dash is a subsequence an element of X dash is also the element of X and X is given to be bounded so the element of the subsequence is also bounded so ,since the X dash is a subsequence of X which is bounded given ,so all the terms of the sequence xn dash is also less than equal to M for all K belongs to N, this criteria will have now this sub sequence is a bounded sequence so X dash itself is a sequence now, this sequence is a bounded sequence once it is a bounded sequence then by Bolzano west's theorem a bounded sequence of real number has a convergent subsequence, convergent sub sequence has a convergent subsequence, ok.

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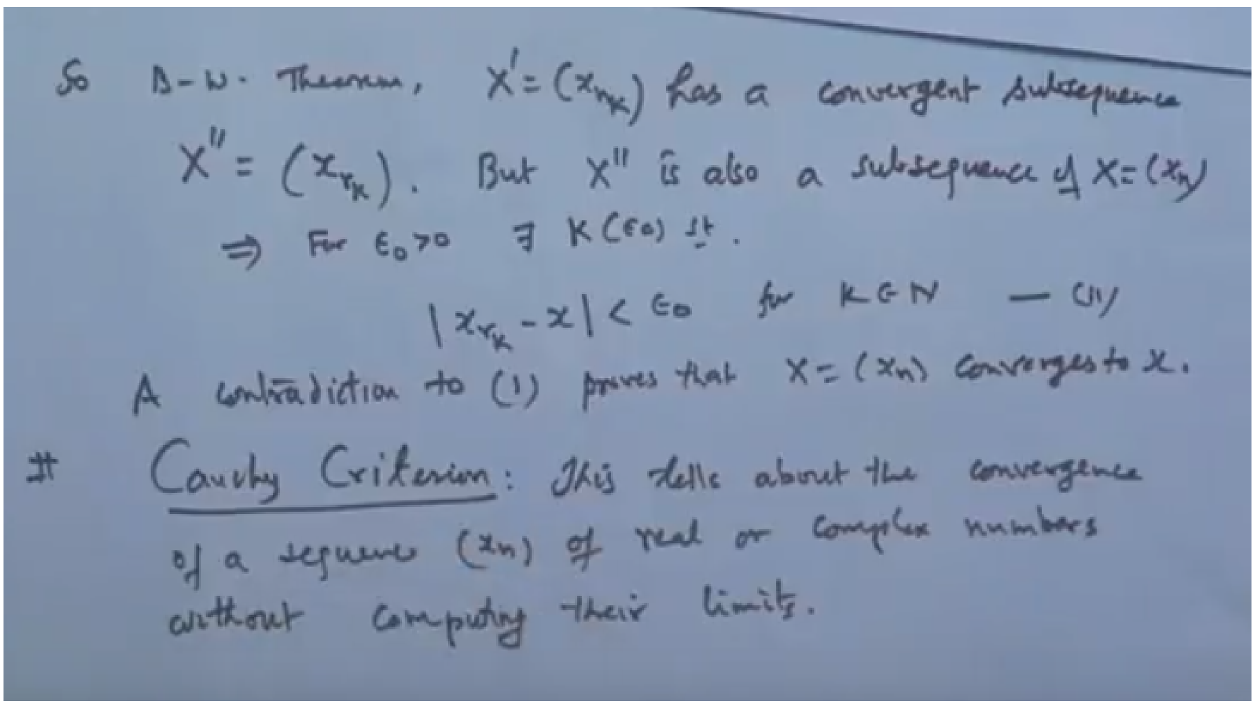

So by Bolzano west's theorem this convergent subsequence, so by Bolzano west's theorem, Bolzano west's theorem we can say that x dash has a convergent, has a convergent, convergent subsequence x double dash, X Double dash say X r k convergent subsequence X double dash okay, okay. but this X double dash is also a subset elements of this are the elements of X but X double dash is also a subsequence of X and this sequence has a subsequence which is convergent, is it not? So this sub sequence converges therefore what we get so by this, so this implies that for a given epsilon naught greater than 0, there exists an integer capital and say K depends on epsilon naught such that mod of x RK minus X, is less than epsilon naught, is it not? But for all k belongs to capital N let it be 2, now first and 2 are contradictory this first is x NK minus X K is greater than epsilon or for now this n K covers all K because all K is one of the in n case because this R this XR k is a sub sequence of xn K so these are the points belongs to X dash so they are also satisfy the condition 1, but this can they also set is they are also satisfying 2 show a contradiction to one, so a contradiction but so a contradiction, contradiction to 1 it proves that so why it is contradiction? Because our assumption is wrong that xn is not convergent prove that sequence xn is convergent, xn converges to X this is what we have, okay. so this proof now let's come to Cauchy convergence criteria, okay. Criteria, now this Cauchy convergence criteria tells, this is this tells about the convergence of a sequence xn of real or complex numbers, without computing their limit, they are limit, limits. because what happen is if the sequence xn is given then one can easily identify and by taking the limit if I consider the limit of the sequence and if the limit comes out to a finite quantity limit exists comes out to a finite quantity then we say the sequence is convergent but if the limit goes to infinity or does not exist means along different subsequence different limits then we say the sequence diverges, but what the Cauchy convergence criteria says that you need not to compute the limits of a sequence xn just simply apply that criteria which is given by Cauchy, one can identify whether the given sequence is a convergent one or diverging or diverging one, okay. so that is the various advantage of this Cauchy convergence criteria because the previous criteria which we have seen whether it is a monotonic convergence theorem or maybe a Bolzano west's theorem, all these theorem depends on certain particular cases, say monotone convergence unless it is sequence is monotone and bounded you cannot say it is a convergent sequence, okay.

Bolzano west's theorem a bounded sequence has a convergent subsequence it does not say about that sequence itself whether the sequence is convergent or not so this Cauchy convergence criteria is very interesting and important and it directly relates to the convergence part of the sequence. So we will go in detail with next class for this okay.

Thank you very much.