Model 3

Lecture – 17

Limit of Sequence and Monotone Sequence

Course

On

Introductory Course in Real Analysis

So this is in continuation of our previous talk. Here we will discuss the limit of sequence, of a real numbers, with few examples. And we will also discuss the monotonic sequence and oscillatory sequences.

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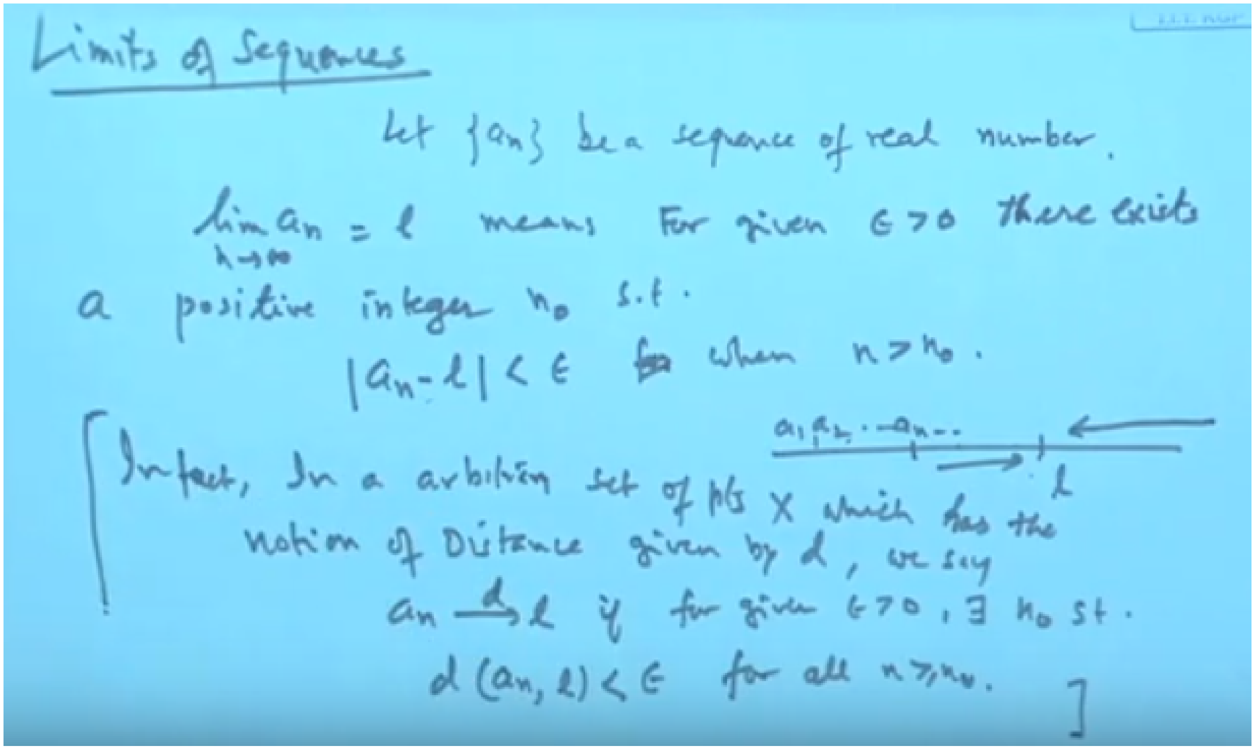

Now we will discuss the limits of sequences. Limit of sequences, Okay? Suppose A and B a sequence. We say, the sequence n is the limit. Let n be a sequence of real number. I am just saying down real number, Okay? I am not, now taking any rational or something, real numbers. Oh yes, Okay, Okay, Okay. Let A and B be a sequence of real numbers, suppose, we say limit of the n, as n tends to infinity, is a L. Means for a given epsilon, greater than 0, for a given epsilon greater than 0, there exist, there exist, a positive integer and not such that, the difference between n minus L, this difference can be made less than epsilon, for all n, when n is greater than, n naught, Okay? The meaning of this is very clear. Suppose we have this number L and there are the sequences a 1, a 2, a n and so on. We say this sequence converges to L. It means, that if we find out the distance, from each L, from each term distance, of each term from L, then this distance, keep on reducing and reduced it to zero. Then we say the sequence n, converges to L. Basically this mode, is the distance. This mode means that absolute difference between the two values.
Now N and L both are real numbers, they can be represented by means of a point, on the real, on the axis. So once you have the point, you can identify the distance and this is the distance a n minus L. Why absolute value, because there may be a sequence, which may converge from this side, it may go from left hand side. So L may be less than this, L may be greater than this. So, but in absolute value, that distance must tends to 0, as in test. So we say the sequence is said to be convergent, when the difference between n minus L or the distance of n from L, keep on introducing and reduce to 0. So this is the way we convert. Now this has been generalized, to an arbitrary metric space, because this is the case, when we are dealing with the real numbers only or complex numbers, then when should, we have the real or complex number, the distance notion, is simply the absolute value. Is it not? T the absolute difference between the two

point, is the distance, of the tuned real number or distance between the two complex number. But suppose X is an arbitrary set of Points, then the notion of the distance will be defined, in such a way, so that the exams of the usual notation, must retain. Okay? That is this.
So that we will take a, infact in an arbitrary space in an arbitrary set of points X, Okay? Which has which has the notion of the distance by D, notion of distance given by D, Okay? given by D, we say a sequence a n, converges to L under D, if, for a given epsilon, greater than 0, there exist an N naught, such that the distance between n, comma L, is less than epsilon, for all and get of than equal to n naught. This is a general way. But we are not dealing with the general, that is why is strictly only mode. But what is the distance function? Distance means, that a distance D, a function D, is a mapping. What is the metric or distance function? This is non-negative, it always you get other than or equal to 0 and 0 when a and the two points are considered. Okay? Then we get the reversal , if n and L in position reverse, we get the same value? and then the tangler inequality. So these conditions are satisfied, then we. So we are not there, that is why we will drop this one and we will simply take up the mode sign, just to say, Okay? Because here, in fact I wanted to introduce that metric distance, but because it is only, so that's why it is not, Okay? So we will take this.

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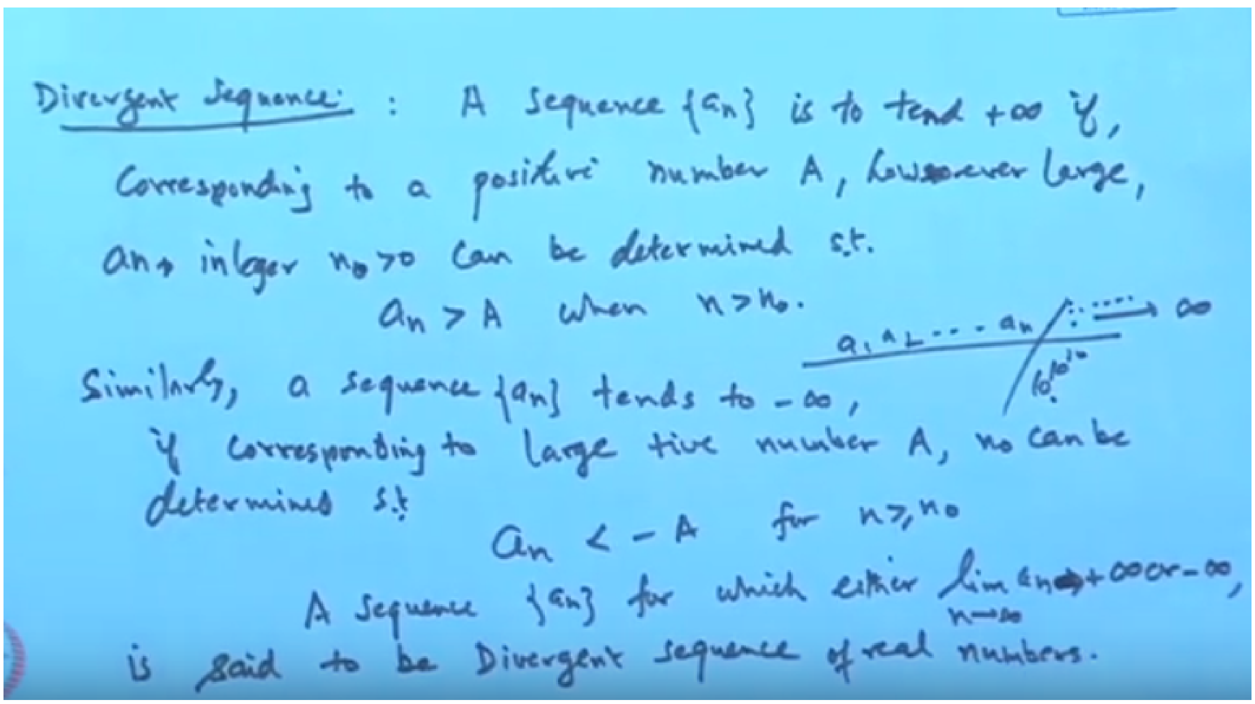

Now, Divergent sequence is, divergent sequence, we define the divergent sequence and is said to be diverging, A sequence, an, is said to tend is sent to plus infinity is sent to plus infinity, if corresponding to a positive number, if corresponding to a positive, corresponding to positive number A, howsoever large ,however large, a number an integer n naught, an integer n naught, greater than 0, can be found, can be determined, such that, an is are greater than A, for whenever n is greater than n naught. We say the sequence n tends to plus Infinity, so these are the sequence a 1, a 2, a n and so on. And this tends to plus Infinity. It means, the limit of the s sequence n, is not finite. It is infinite. So it is a diverging sequence. A sequence is said to be diverging, when the limit of the sequence does not exist. Either it will be a plus infinity or minus infinity, then it is said to be a diverging sequence. Okay? So when you say it is a sequence a n, goes to plus infinity means, that whatever the number you choose, you can always find an integer n naught, such that the value of the coordinate of the sequence, will exceed by that number. Suppose I would say, a is equal to 10 to the power 10, to the power 10, then this number is there, say 10 to the power 10, to the power 10, like this. Then one can identify a number n naught here. That all the terms of the sequence, after this, will greater than this number. So we say it is tending to plus infinity. Similarly we say, a sequence, a sequence a n, similar is you can tends to minus infinity, if corresponding to, if corresponding to, a large positive number, a large positive number a, n naught can be determined, such that a n's are less than minus a, for all n, greater than equal to n naught. Then it is tending to minus infinity, Okay?

So a sequence which is either so, a sequence a n, for which, either the limit of a n, as n tends to infinity, is plus infinity or minus infinity, limit of this tends to, not tends to, plus infinity or minus infinity or minus infinity, is it not plus infinity or minus infinity is said to be, is said to be, diverging sequence. Sequence of real numbers, Okay? Of real numbers ,Okay? For examples, suppose I take a n, the sequence say, n square, this will diverge? Okay, similarly other sequences also you can say, it will diverge to plus infinity, if I take a n, equivalent to say minus n, it will diverge to minus infinity, like this and so on so, Okay?

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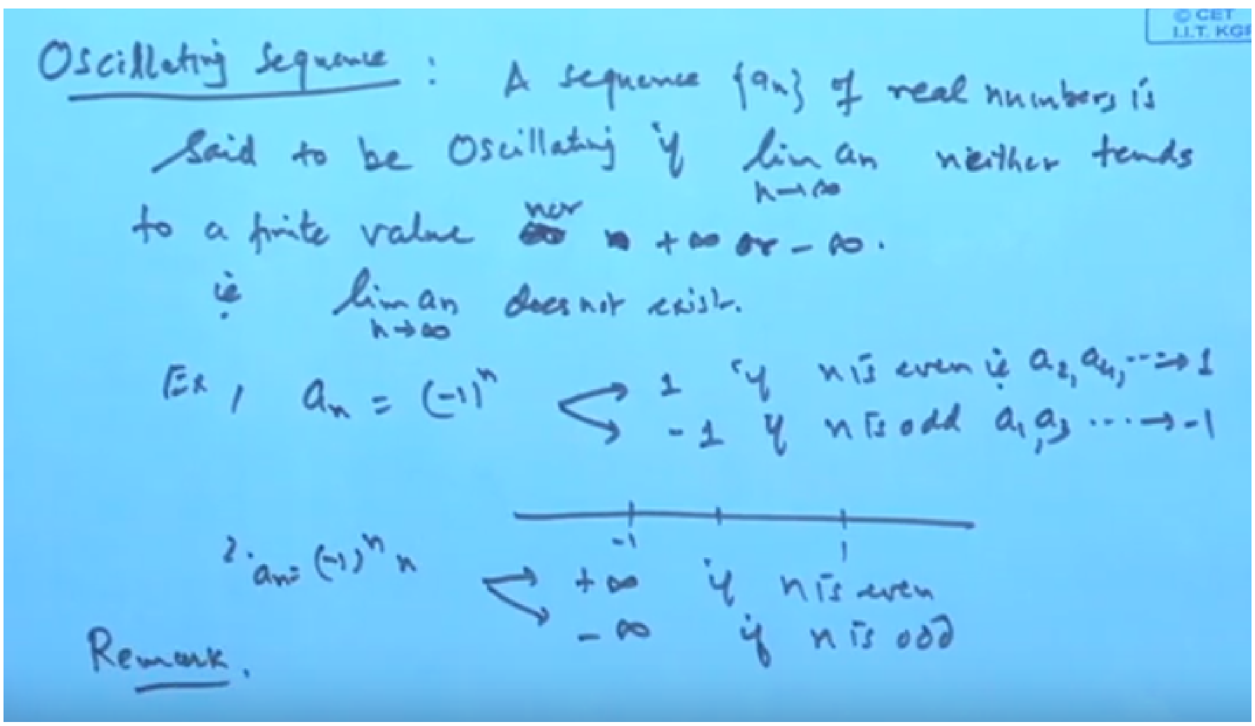

Then Oscillating sequence , Oscillatory, Oscillatory sequence, a sequence a n, of real numbers, is said to be, oscillatory is said to be oscillating, if if the limit of n, as n tends to infinity, does not exist and limit of this neither, does not exist or neither, neither tends to a finite value, neither tends to a finite value or all nor all, plus infinity or minus nl plus infinity or minus infinity, plus infinity or minus. The meaning is that, n is such, where the limit does not exist. Limit does not exist, means that is limit of the sequence, n when n tends to infinity, does not exist. Because when we say the limit exists it means, whatever the path you choose, because the definition of the limit, when the limit is there, this is the definition of limit, that if the limit exists means, this is less than so what table the path you choose limit of n minus L can be made as small as we please.

The difference between a and L should be made a smaller, in this edge. One can decide, desire. There should not be fluctuation. But if such a sequence are there, whether this difference cannot be made a smaller, sometimes it is small, sometimes become very large, then in that case the limit does not exist or along different sub sequences it has different values, then the limit does not exist. For example, if suppose I take a n, to be minus 1, to the power n. Then along the positive Path, it well, it will go to 1, if n is even. That is when the sequence are chosen like this, a 2, a 4, etcetera, the limit will go to 1. But if the sequence is choosen to be odd, then n is odd. That is, a 1, a 3, this limit will go to minus 1. So the sequence when n tends to infinity, does not tends to a, 1 value. Because, it fluctuate like this. Here is minus 1, here is plus 1. So what happen is, a n, you cannot find any, suppose, I take 1 is the value, then a n minus 1, cannot be made s1, because S1 has a n or becomes then it will be minus 1, minus 1, minus 2, so it becomes very large. Similarly here. So it does not go to that, does not have a finite value, limit. Similarly if you take the sequence minus 1, to the power n, n, say this one. Then what happens? When n is sufficiently large, for an even Number, it will go to plus infinity, it goes to plus infinity, if n is even and goes to minus infinity, if n is odd. So it does not have the limit. Okay? Similarly.

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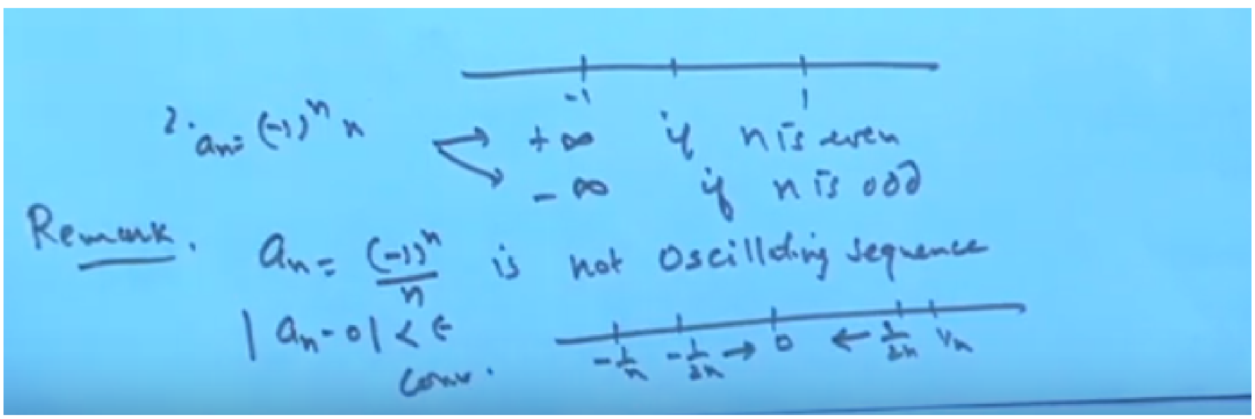

However the sequence a nm which is minus 1 to the power n by n,is not an oscillating series, is not the C, is not a sequence, Sorry, is not an oscillatoring sequence. Why? Oscillating sequence of those sequence, with limit, does not exist either, the limit is not tending to a finite value, or plus infinity or minus infinity. Now this sequence, tends to value 0, though it is alternately positive negative, but what happens? If this is the value 0, you are getting minus 1 by n, plus 1 by n, then as n increases, you are taking minus 2 by n, plus 1 by 2 n, like this. So this goes to here, this goes to here. So after certain state, the difference between a n, minus 0, can be made as small, as we please. Is it not? So that is why, this sequence is a convergent sequence, converges to 0. So it is not zero, but convergent, Okay? This is convergent.

Then Bounded Sequence, so we say, the, here also we can characterize the oscillating sequence, to finitely oscillating, and infinite. Finitely Oscillating sequence, when the limit tends to, it does not tends to finite value, but does not go to plus infinity, minus infinity, but it alternates. And in finite means, that is when we are unable to get, that is a sequence is said to be a finitely order and no remark.

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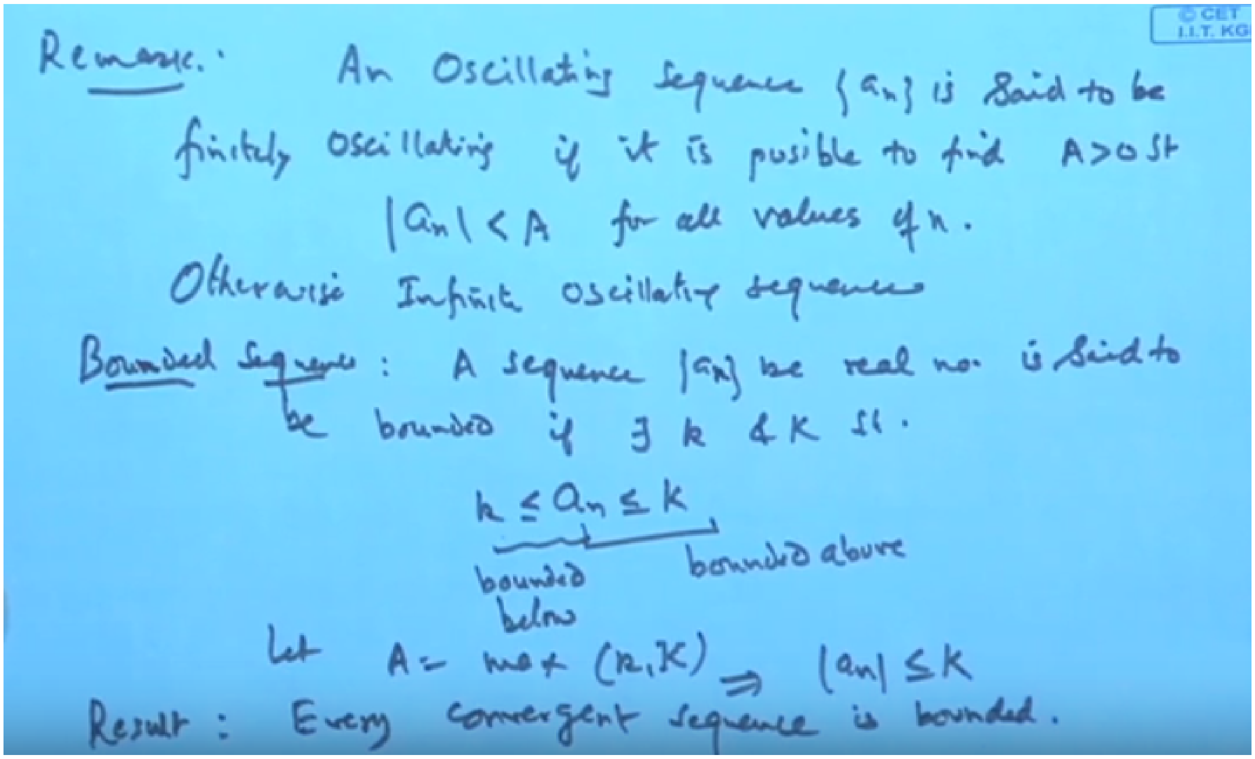

A sequence, an oscillating sequence, sequence an, is said to be, is said to be finitely oscillating, if, if there exist number a, if it is possible to find a number a, greater than zero, such that, all the terms of the sequence remain less than a, all the terms of the sequence remain less than a, for all values of n. We are able to get it, just like this. See. This sequence is a finitely Oscillatory, because a number one, can be Such that the mode of this n. But this is not a. Because it does not even a,you cannot find. Otherwise, otherwise, infinite oscillatory sequence, Okay? So we can characterize these two into two, Okay.

Bounded sequence, we have already discussed, so no point of, a bounded sequence. And then, every convergence sequence is bounded. A sequence a n, is said to be bounded, a sequence n of real numbers, is said to be bounded, if there exists k and capital K, such that, a n is greater than equal to a small k, greater than equal to, is less than equal to capital K. Suppose we have the two bond. It is not Necessary, that we have the same bound, Okay? So a sequence n, is said to be bounded below, if there is a small k, such that all the terms of the sequence are greater than equal to K, then this is, this will give the bounded below, bounded below, well this thing will give bounded above. Okay? Bounded above. And if we combine both these and let capital A, be the maximum of k and capital K, Okay? In fact this is capital K only, Okay? Then in that case, mode of n, is less than equal to K. Then we say this sequence is bounded. Okay? Bounded. So lower bounded, above bounded and bounded or like this, Okay? So we get this one. And every convergent sequence, is a bounded sequence. That we have seen in the. So result, is every convergent sequence, is a bounded, is bounded sequence, is bounded. I think this proof we have done is, what about the converse? Can you say every bounded sequence, is convergent.

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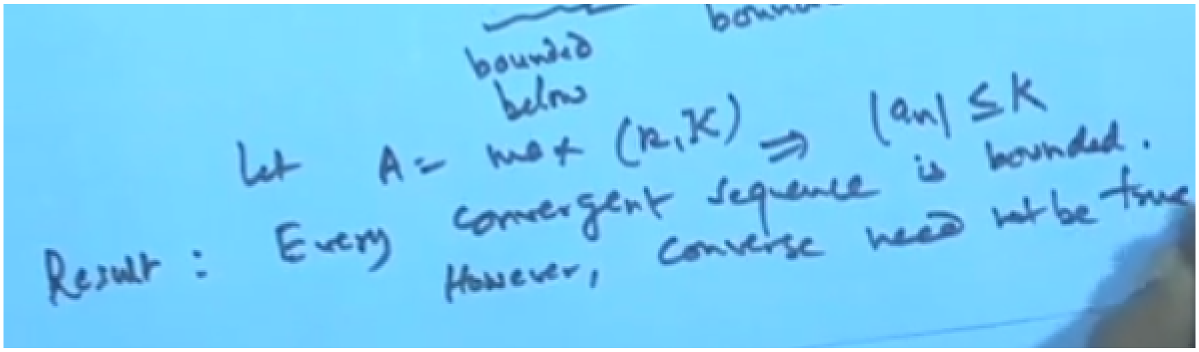

The answer is,' No'. But converse is not true. However converse need not be true.

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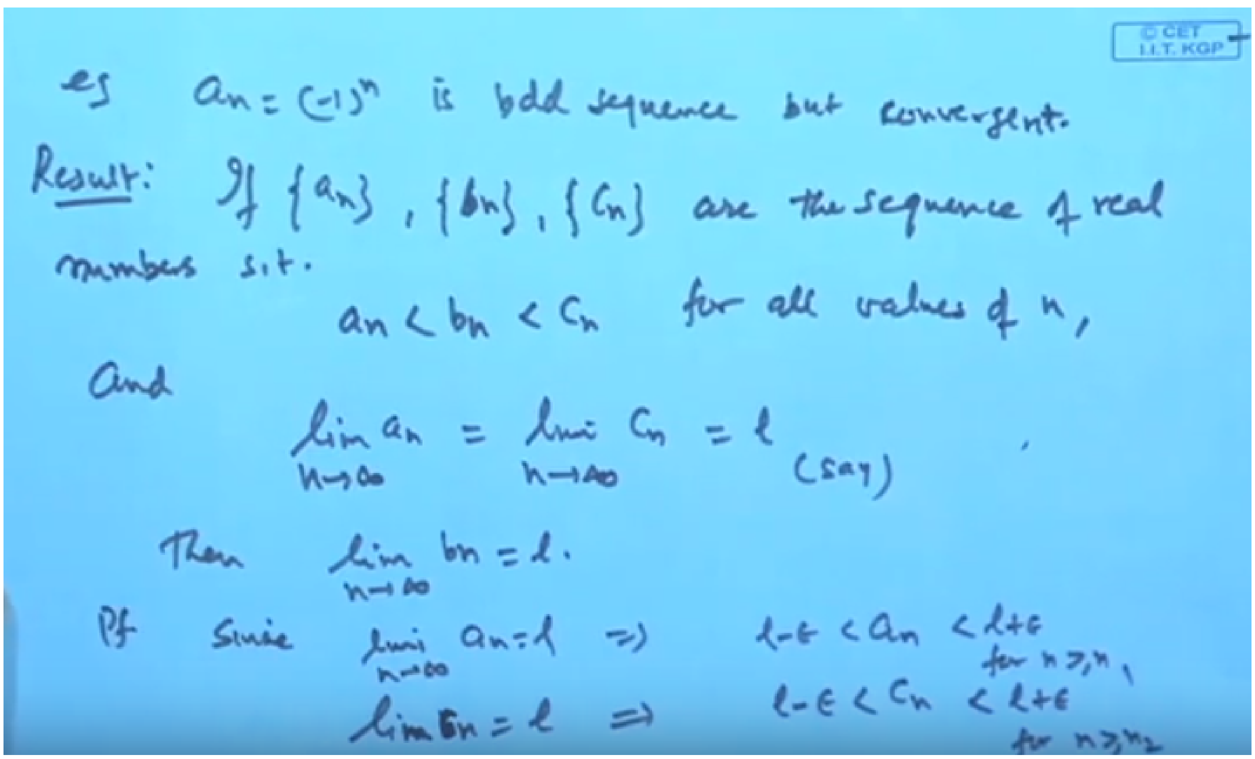

Because if you take that for example. If you take the, yes, if you take the sequence a n, minus 1, to the power n, this is a bounded sequence, but not convergent, Okay? So we can get this one, clear? Now fundamental theorems of this limit, is the same as this, so we are not touching those things, Okay, let us see the results, which is very important result.

There is one result, which is known as the, 'Sandwich theorem', sandwich theorem. What does this says is, if sequence an, BN and sequence c n. Let an, BN, and CN, be the, are the sequences such that of real numbers, such that, are the sequences of real numbers, such that, a n, are less than B n, less than C n, Okay? Suppose we have this sequence and this is to for all values of n, values of n. That is three sequences are given and they satisfy this inequality for all N. And limit of n, as n tends to infinity, is the same as the limit of CN, as n tends to infinity, and suppose it is L. Then what this result says, then the limit of this sequence BN, will also be L. This is known as the sandwich theorem. That, if you want to find the, limit of the sequence be B N, if we are able to identify that Low l and the upper bounds, for each n, that is a a n sequences, where the corresponding terms are satisfying this condition and if this the left-hand sequence and right rain sequence converges to the same limit, then the middle sequence will also converge to the same, Okay? Okay? The proof is very simple, proof is not that. Because, proof is why it is so? Because n is given to be L. Okay? So since limit of this an, is L. So it implies that a n, must lie between L ,minus epsilon L and L plus epsilon, for n greater than equal to n1. Similarly limit of this VN, is L. So this implies that, L minus, say same epsilon we can choose all different also, Cn, sorry, limit of Cn. So L minus Epsilon, less than CL, less than L plus Epsilon ,after integer and greater than equal to n2. Now if I picked up the n greater than n1 and n2.

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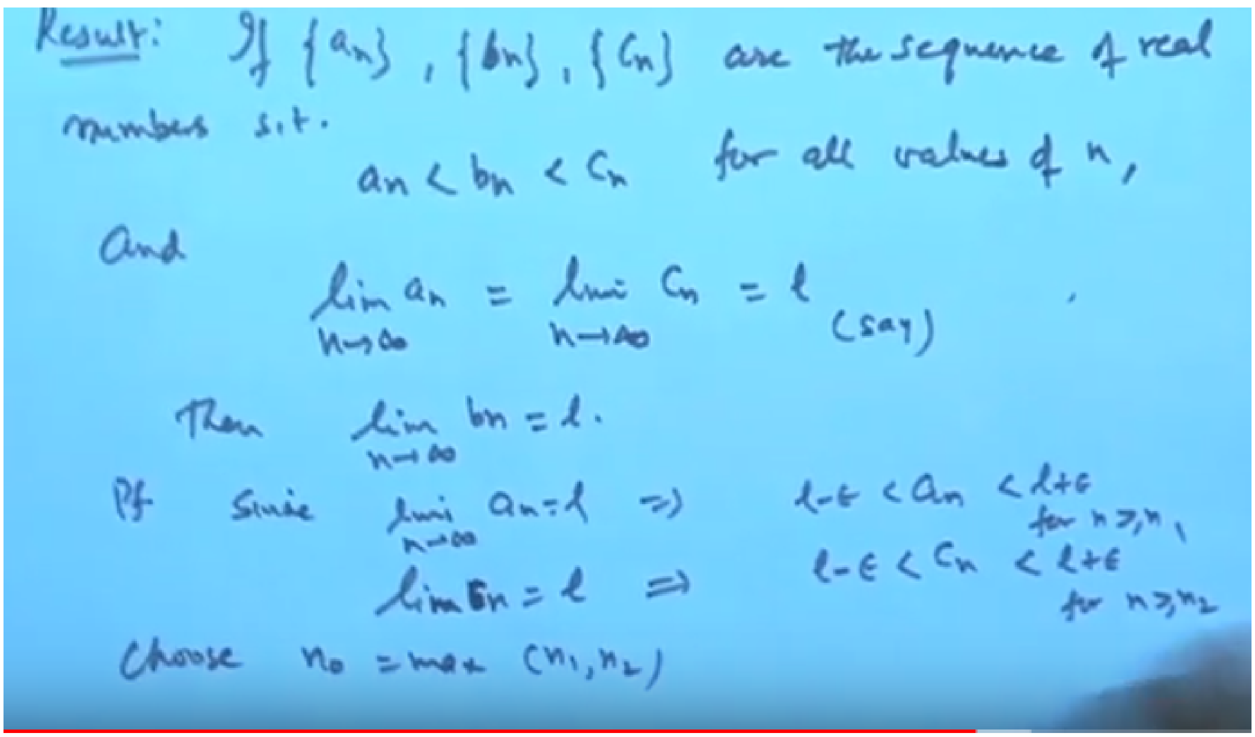

Choose n naught, as the maximum of N 1 and n 2. So for all N, greater than n naught, this condition is satisfied, for all n greater than n2, this condition will satisfy, Okay? So, we get from here, an and BN lying, so BN is the number lying Between AN and C n, so we can say,

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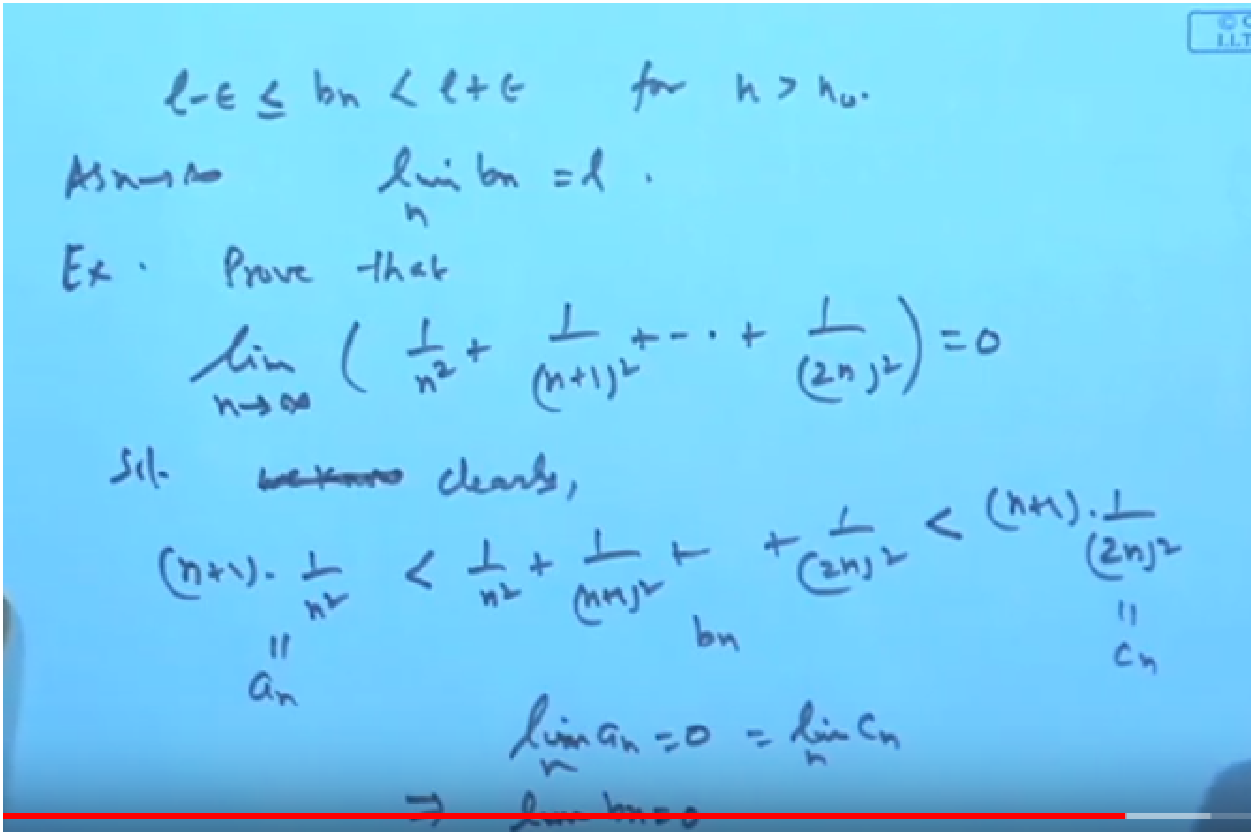

L minus Epsilon, is less than B n, is less than B N, at the most equal to, less than L plus epsilon, for all N, greater than n naught. Is it not? Clear? So as n tends to Infinity, limit of the BN, will go to L. So this point. Okay? Now this will this is used, to find the limit of the complicated expression. For example, what is the use, suppose I take, this problem. Prove that, limit of this, as n tends to infinity, 1 by n square, 1 by n plus 1 whole square, and so on, 1 by 2 n square, is 0. Limit of this is 0, we wanted to show this part. So we know, we know, that 1 by n square, 1 plus, the lowest term is 1 by 2 n? Up and largest term is this? So the calculation shows, that 1 by n Square, 1 over n plus 1, whole square and 1 by 2 n square. This will be less than, total terms are what? n plus 1, is starting with n 2, n plus 1, so total term is n plus 1, into 1 by 2 n square and greater, then n plus 1 into 1 by n square? Okay? Or clearly it just, we get this? Is it not? Now as n tends to infinity, this is our Ns' this is our Cns'. So as n tends to infinity, because the denominator is having larger degree than the denominator, so this will go to 0, this will go to 0. So AN and CN, limit of this a n is 0, is the same as the limit of C n. Therefore limit of BN, must go to 0. So this implies limit of this BN, must. This is about BNs', Okay? So this shows the very interesting things. Is it? Like this.

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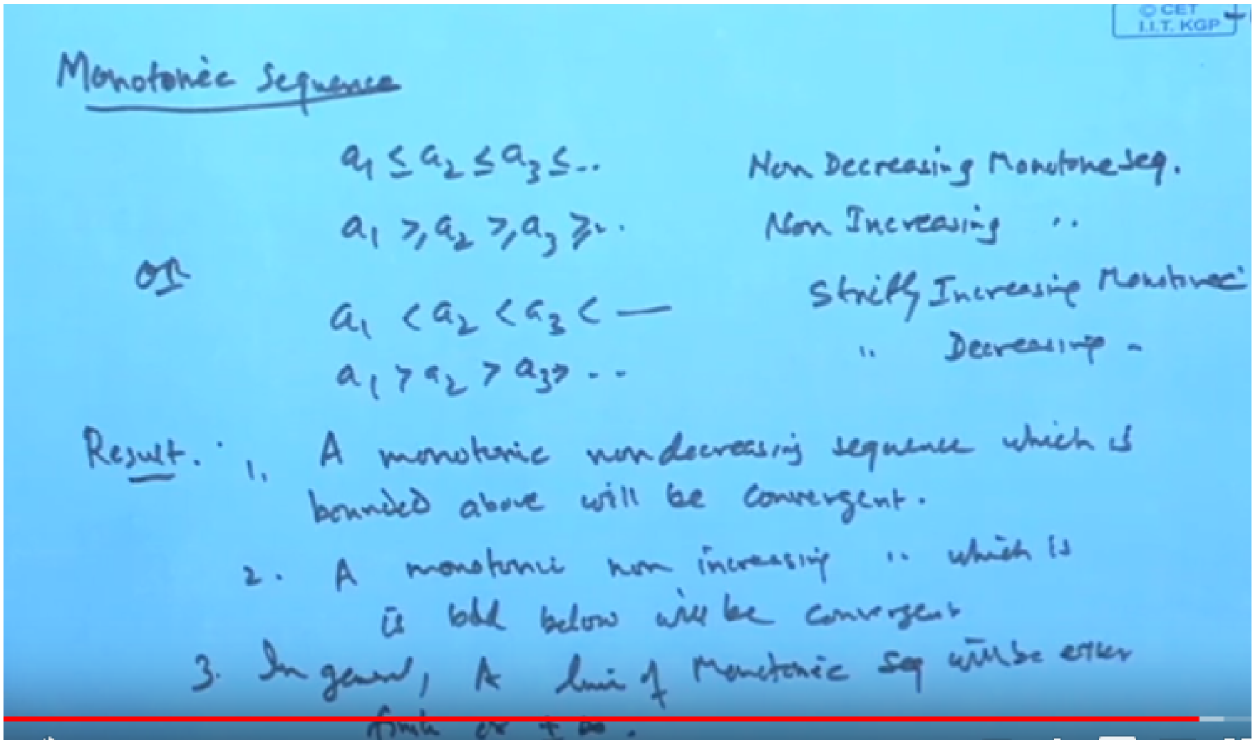

Monotonic sequence, there are also Similar, similar types of this monotonic, monotonic sequence, sequences. We have seen that, there are two types of sequences, which are either in, non decreasing or in. Means a 1, is less than equal to a 2, less than equal to a 3, etcetera. These are called the non decreasing, sequence, monotone sequences. Non decreasing. Okay? It keep on, but may be constant also. Or a 1, greater than equal to, a 3, this is called, non increasing sequence, monotone sequence. Or if we have this one, then it is called a strictly increasing monotone sequences. Okay? Monotonic. And if we have this one, then we say, is strictly decreasing. So we get this one. Okay? Now if this monotone sequence are there, and if it is bounded above, say, non decreasing sequence which is bounded above, then it will have the limit. If it is a monitor and decreases taking bounded below, then it will also have a limit. So these two results, we have already discussed it. Is it not? So just I will just result, Okay, the, a monotonic decreasing sequence, a monotonic increase sequence, tends to either to limit or. A monotonic, increasing, non-decreasing sequence, non-decreasing sequence, which is bounded above, bounded above, will be convergent. Will be convergent. Similarly a monotonic non increasing sequence, which is bounded below, which is bounded below, will be convergent.

In general, a monotone sequence, the limit of the monotone sequence, will be either finite or plus, minus, infinity, in general. But these are the convergent credit. So either monotinic, in decreasing, non decreasing or strictly increasing or strictly increasing, or is strictly decreasing, that is what, this also leave it. Now this will be used also, to find the Limits.

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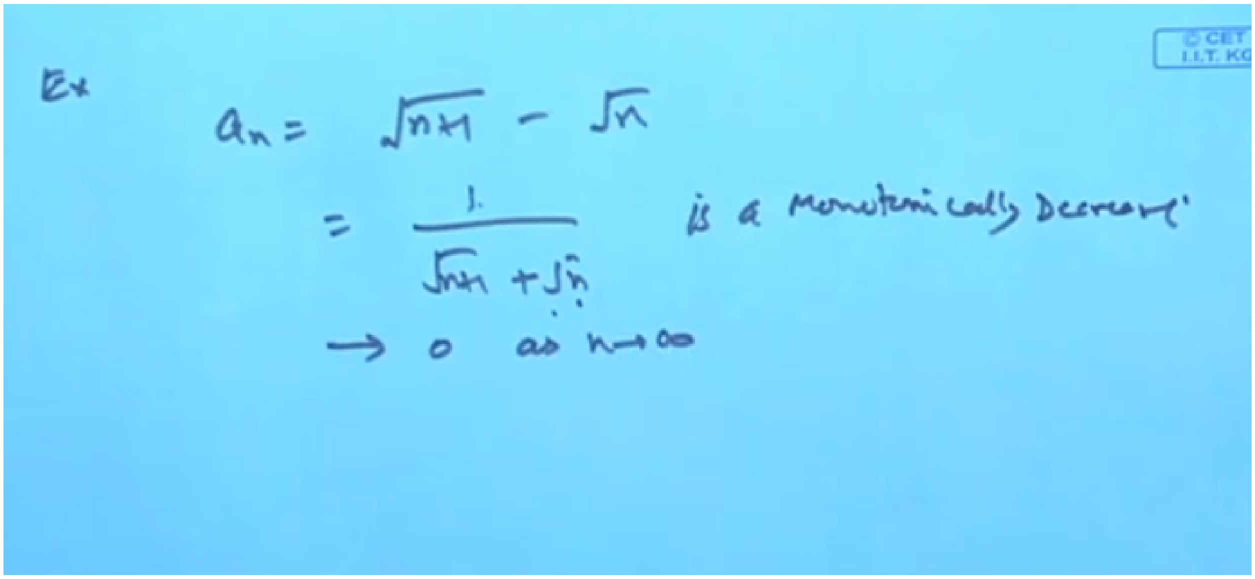

Suppose for example, if we take this a n, sequence, edge under root, n plus 1, minus under root N. And I ask what is the limit, of this? So if you go to the limit as n tends to infinity, what you are getting? Infinity minus infinity? Which in the in determinant. We cannot get it. But if you slightly if you manipulate it, we get divide and multiply by this. So when you multiply by this number, then you get aX square minus aX square, is that becomes one. And this is equivalent to this sequence. Is it not? Now this sequence is monotonically, is a monotonically, what? Increasing or decreasing? Decreasing, and tends to 0 as n tends to. So limited 0. That is what. Is it not? So we will look that sub limits also, which are very interesting, particularly the 2, 3 limits which we get as an exercise next time.

Okay. Thank you very much