Lecture 04: Behavior of Regular Sturm–Liouville Systems Part II

1 The Sturm-Liouville Operator

Let $I = (a, b)$ be a finite interval. We define the Sturm–Liouville operator

$$L[y] := (p(x)y'(x))' + q(x)y(x),$$

where the coefficient functions satisfy

$$p, q, s \in C^\infty([a, b]), \qquad p(x) > 0, \quad s(x) > 0 \text{ on } [a, b].$$

For simplicity, we take the domain of L as

$$D(L) = C^\infty([a, b]),$$

though in general $C^2([a, b])$ suffices.

The associated eigenvalue problem is

$$L[y] + \lambda s(x)y = 0, \tag{1}$$

together with regular Sturm–Liouville boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0,$$

where $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$.

2 Orthogonality of Eigenfunctions

If (λ_k, ϕ_k) and $(\lambda_\ell, \phi_\ell)$ are two distinct eigenpairs with

$$\lambda_k \neq \lambda_\ell,$$

then the corresponding eigenfunctions are orthogonal with respect to the weight $s(x)$:

$$\int_a^b s(x)\phi_k(x)\phi_\ell(x)\, dx = 0.$$

3 Lagrange Identity

Theorem 1 (Lagrange Identity)**.** *Let $y, z \in D(L)$. Then*

$$yL[z] - zL[y] = \frac{d}{dx}\Big(p(x)\big(yz' - zy'\big)\Big).$$

Proof. A direct computation shows

$$y(pz')' - z(py')' = \frac{d}{dx}\big(p(yz' - zy')\big),$$

and the qyz terms cancel. □

4 Self-Adjointness of the Sturm-Liouville Operator

We define the L^2 inner product

$$\langle f, g\rangle := \int_a^b f(x)\overline{g(x)}\,dx.$$

Theorem 2. *The Sturm–Liouville operator L is self-adjoint, i.e.*

$$\langle L[y], z\rangle = \langle y, L[z]\rangle \quad \textit{for all } y, z \in D(L).$$

Proof. Using the Lagrange identity,

$$\langle L[y], z\rangle - \langle y, L[z]\rangle = \Big[p(x)(\overline{z}y' - y\overline{z}')\Big]_a^b.$$

The boundary conditions imply that this boundary term vanishes at both a and b. Hence the operator is self-adjoint. □

5 Reality of Eigenvalues

Theorem 3. *All eigenvalues of a regular Sturm-Liouville boundary value problem are real.*

Proof. Let (λ, y) be an eigenpair. Since L is self-adjoint,

$$\langle L[y], y\rangle = \langle y, L[y]\rangle.$$

Using the eigenvalue equation,

$$\lambda \int_a^b s(x)|y|^2\,dx = \overline{\lambda} \int_a^b s(x)|y|^2\,dx.$$

Since $s(x) > 0$ and $y \not\equiv 0$, we conclude $\lambda = \overline{\lambda}$. □

6 Abel's Formula

Theorem 4 (Abel's Formula)**.** *Let* ϕ_1 *and* ϕ_2 *be two solutions of* (1) *corresponding to the same* λ*. Then*

$$p(x)W(\phi_1, \phi_2)(x) = \textit{constant},$$

where

$$W(\phi_1, \phi_2)(x) := \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x).$$

Proof. Multiply the equation for ϕ_1 by ϕ_2 and that for ϕ_2 by ϕ_1. Subtracting yields

$$\frac{d}{dx}\big(p(x)W(\phi_1, \phi_2)(x)\big) = 0.$$

Hence the expression is constant. □

7 Uniqueness of Eigenfunctions

Theorem 5. *For a given eigenvalue* λ*, any two eigenfunctions are linearly dependent.*

Proof. Let ϕ_1 and ϕ_2 be eigenfunctions corresponding to λ. Both satisfy the boundary conditions, which imply

$$W(\phi_1, \phi_2)(a) = 0.$$

By Abel's formula, $p(x)W(\phi_1, \phi_2)(x)$ is constant, hence identically zero. Thus $W(\phi_1, \phi_2) \equiv 0$, which implies linear dependence:

$$\phi_1 = C\phi_2.$$

□

Summary

- Distinct eigenfunctions are orthogonal with respect to $s(x)$.
- The Sturm–Liouville operator is self-adjoint.
- All eigenvalues are real.
- For a fixed eigenvalue, eigenfunctions are unique up to a multiplicative constant.