Lecture 04: Behavior of Regular Sturm–Liouville Systems
Part A

1 Regular Sturm–Liouville Problem

In this lecture, we study some fundamental properties of a *regular Sturm–Liouville boundary value problem*. The differential equation under consideration is

$$\big(P(x)y'(x)\big)' + Q(x)y(x) + \lambda S(x)y(x) = 0, \tag{1}$$

defined on the interval

$$I = [a, b].$$

Throughout this lecture, we assume that

$$P, Q, S \in C^\infty([a, b]), \qquad S(x) > 0 \text{ on } [a, b],$$

and that $\lambda \in \mathbb{R}$.

The associated boundary conditions are of the form

$$\begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0, \\ \beta_1 y(b) + \beta_2 y'(b) = 0, \end{cases} \tag{2}$$

where

$$\alpha_1^2 + \alpha_2^2 \neq 0, \qquad \beta_1^2 + \beta_2^2 \neq 0.$$

The problem (1)–(2) is called a *regular Sturm–Liouville problem.*

2 Orthogonal Functions

Before studying eigenfunctions of (1), we introduce the notion of orthogonality for functions.

Definition 1 (Orthogonal Functions)**.** *Let ϕ and ψ be real-valued integrable functions on $[a, b]$, and let $\rho(x) > 0$ be a given function on $[a, b]$. We say that ϕ and ψ are* orthogonal with respect to the weight function ρ *if*

$$\int_a^b \phi(x)\psi(x)\rho(x)\, dx = 0.$$

The function ρ is called the *weight function.* For example, when $\rho(x) \equiv 1$ on $[-\pi, \pi]$, the trigonometric functions $\sin(nx)$ and $\cos(mx)$ are orthogonal whenever $n \neq m$.

3 Norm of a Function

Just as vectors have lengths, functions have norms.

Definition 2 (Norm)**.** *Let ϕ be a real-valued integrable function on $[a, b]$ and let $\rho(x) > 0$. The norm of ϕ with respect to ρ is defined by*

$$\|\phi\| = \left(\int_a^b \phi^2(x)\rho(x)\, dx \right)^{1/2}.$$

4 Orthogonality of Eigenfunctions

We now establish a fundamental result concerning eigenfunctions of regular Sturm–Liouville problems.

Theorem 1. *Let (λ_j, ϕ_j) and (λ_k, ϕ_k) be two distinct eigenpairs of the Sturm–Liouville problem* (1)–(2)*. Then the eigenfunctions ϕ_j and ϕ_k are orthogonal with respect to the weight function $S(x)$ on $[a, b]$, i.e.,*

$$\int_a^b \phi_j(x)\phi_k(x)S(x)\, dx = 0.$$

5 Proof of the Theorem

Proof. Since ϕ_j and ϕ_k are eigenfunctions, they satisfy

$$(P\phi_j')' + Q\phi_j + \lambda_j S\phi_j = 0,$$

$$(P\phi_k')' + Q\phi_k + \lambda_k S\phi_k = 0.$$

Multiply the first equation by ϕ_k and the second equation by ϕ_j, and subtract:

$$\phi_k(P\phi_j')' - \phi_j(P\phi_k')' + (\lambda_j - \lambda_k)S\phi_j\phi_k = 0.$$

The Q-terms cancel. Observe that

$$\phi_k(P\phi_j')' - \phi_j(P\phi_k')' = \frac{d}{dx}\Big(P(\phi_k\phi_j' - \phi_j\phi_k')\Big).$$

Integrating over $[a, b]$ yields

$$(\lambda_j - \lambda_k)\int_a^b S\phi_j\phi_k\, dx = \Big[P(\phi_k\phi_j' - \phi_j\phi_k')\Big]_a^b.$$

We now use the boundary conditions. Since both ϕ_j and ϕ_k satisfy (2), one can verify that

$$\phi_k(b)\phi_j'(b) - \phi_j(b)\phi_k'(b) = 0,$$

and similarly,

$$\phi_k(a)\phi_j'(a) - \phi_j(a)\phi_k'(a) = 0.$$

Hence, the boundary terms vanish, and we obtain

$$(\lambda_j - \lambda_k)\int_a^b S(x)\phi_j(x)\phi_k(x)\, dx = 0.$$

Since $\lambda_j \neq \lambda_k$, it follows that

$$\int_a^b S(x)\phi_j(x)\phi_k(x)\,dx = 0.$$

Therefore, ϕ_j and ϕ_k are orthogonal with respect to the weight function $S(x)$. $\square$

Conclusion

Distinct eigenfunctions of a regular Sturm–Liouville problem are orthogonal with respect to the weight function $S(x)$. This orthogonality property plays a central role in Fourier–Sturm–Liouville expansions and in solving partial differential equations.