

# Lecture 03: Sturm–Liouville Problem

## An Introduction

### 1 Introduction and Motivation

In the previous lectures, we studied Fourier series. Sturm–Liouville theory generalizes Fourier series and plays a central role in the theory of partial differential equations (PDEs). It allows us to study families of second-order ordinary differential equations and associated eigenvalue problems.

### 2 General Second-Order Differential Equation

Let  $\lambda \in \mathbb{C}$  be a parameter. Consider the differential equation

$$a_1(x)y''(x) + a_2(x)y'(x) + a_3(x)y(x) + \lambda y(x) = 0, \quad (1)$$

where

$$a_1, a_2, a_3 \in C^\infty(I), \quad I = [a, b],$$

and we assume

$$a_1(x) \neq 0 \quad \text{for all } x \in I.$$

### 3 Self-Adjoint Form

Define

$$P(x) = \exp\left(\int \frac{a_2(x)}{a_1(x)} dx\right), \quad Q(x) = \frac{a_3(x)}{a_1(x)}P(x), \quad S(x) = \frac{P(x)}{a_1(x)}.$$

Then equation (1) can be rewritten as

$$(P(x)y'(x))' + Q(x)y(x) + \lambda S(x)y(x) = 0. \quad (2)$$

### 4 Sturm–Liouville Operator

Define the *Sturm–Liouville operator*

$$L[y] = (P(x)y'(x))' + Q(x)y(x).$$

Then equation (2) becomes

$$L[y] + \lambda S(x)y = 0.$$

## 5 Regular Sturm–Liouville Problem

A regular Sturm–Liouville problem consists of

$$L[y] + \lambda S(x)y = 0, \quad x \in (a, b), \quad (3)$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad (4)$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0, \quad (5)$$

where

$$\alpha_1^2 + \alpha_2^2 \neq 0, \quad \beta_1^2 + \beta_2^2 \neq 0.$$

## 6 Eigenvalues and Eigenfunctions

The trivial solution  $y \equiv 0$  exists for all  $\lambda$ . We seek values of  $\lambda$  for which a *non-trivial* solution exists.

A value  $\lambda$  is called an *eigenvalue* if the Sturm–Liouville problem admits a non-zero solution. The corresponding solution is called an *eigenfunction*.

## 7 Example

Consider

$$y'' + \lambda y = 0, \quad x \in (0, \pi), \quad (6)$$

with boundary conditions

$$y(0) = 0, \quad y'(\pi) = 0.$$

### 7.1 Case $\lambda < 0$

Let  $\lambda = -\mu^2$ ,  $\mu > 0$ . Then

$$y(x) = Ae^{\mu x} + Be^{-\mu x}.$$

Boundary conditions imply  $A = B = 0$ . Hence,  $\lambda < 0$  is not an eigenvalue.

### 7.2 Case $\lambda = 0$

Then

$$y(x) = Ax + B.$$

Boundary conditions imply  $A = B = 0$ . Thus,  $\lambda = 0$  is not an eigenvalue.

### 7.3 Case $\lambda > 0$

Let  $\lambda = \mu^2$ ,  $\mu > 0$ . Then

$$y(x) = A \cos(\mu x) + B \sin(\mu x).$$

From  $y(0) = 0$ , we get  $A = 0$ . From  $y'(\pi) = 0$ , we obtain

$$\cos(\mu\pi) = 0.$$

Thus,

$$\mu = \frac{2n-1}{2}, \quad n \in \mathbb{N}.$$

## Eigenpairs

The eigenvalues are

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2,$$

and the corresponding eigenfunctions are

$$y_n(x) = \sin\left(\frac{2n-1}{2}x\right).$$

## 8 Conclusion

S Sturm–Liouville problems lead to discrete eigenvalues and orthogonal eigenfunctions, forming the foundation of Fourier series methods used extensively in PDE theory.