Lecture 02 : Part C
Addendum: Double Fourier Series

1. Recall: Fourier Series in One Variable

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous and 2π–periodic function. Then f admits a Fourier series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

The Fourier series converges to f in the mean. If $f \in C^1$, then the convergence is uniform.

2. Two–Dimensional Periodic Functions

Definition. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called 2π*–periodic* if

$$f(x + 2\pi, y) = f(x, y), \qquad f(x, y + 2\pi) = f(x, y)$$

for all $(x, y) \in \mathbb{R}^2$.

Example.

$$f(x, y) = \sin x \cos y$$

is 2π–periodic in both variables.

3. Fourier Expansion with Respect to One Variable

Assume $f(x, y)$ is 2π–periodic and continuously differentiable in both variables. Fix y and define

$$\phi_y(x) = f(x, y).$$

Then ϕ_y is a 2π–periodic function of x and admits the Fourier expansion

$$f(x, y) = a_0(y) + \sum_{n=1}^{\infty} \left(a_n(y) \cos(nx) + b_n(y) \sin(nx)\right),$$

where

$$a_n(y) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x, y) \cos(nx)\, dx, \quad b_n(y) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x, y) \sin(nx)\, dx.$$

4. Fourier Expansion of the Coefficients

Since $a_n(y)$ and $b_n(y)$ are 2π–periodic in y, they admit Fourier series:

$$a_n(y) = a_{n0} + \sum_{m=1}^{\infty} \left(a_{nm}\cos(my) + b_{nm}\sin(my)\right),$$

$$b_n(y) = c_{n0} + \sum_{m=1}^{\infty} \left(c_{nm}\cos(my) + d_{nm}\sin(my)\right).$$

5. Double Fourier Series

Substituting these expansions yields the *double Fourier series*

$$\begin{aligned} f(x,y) = a_{00} &+ \sum_{m=1}^{\infty} \left(a_{0m}\cos(my) + b_{0m}\sin(my)\right) \\ &+ \sum_{n=1}^{\infty} \left(a_{n0}\cos(nx) + c_{n0}\sin(nx)\right) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Big(a_{nm}\cos(nx)\cos(my) + b_{nm}\cos(nx)\sin(my) \\ &\qquad + c_{nm}\sin(nx)\cos(my) + d_{nm}\sin(nx)\sin(my)\Big). \end{aligned}$$

6. Fourier Coefficients

The coefficients are given by

$$a_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos(nx)\cos(my)\,dx\,dy,$$

$$b_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos(nx)\sin(my)\,dx\,dy,$$

$$c_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin(nx)\cos(my)\,dx\,dy,$$

$$d_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin(nx)\sin(my)\,dx\,dy.$$

7. Symmetry and Simplifications

- If f is even in both variables, only $\cos(nx)\cos(my)$ terms appear.
- If f is even in x and odd in y, only $\cos(nx)\sin(my)$ terms appear.
- If f is odd in both variables, only $\sin(nx)\sin(my)$ terms appear.

8. Example: $f(x,y) = xy$

Consider

$$f(x,y) = xy \quad \text{on } [-\pi,\pi]\times[-\pi,\pi],$$

extended 2π-periodically.

Since

$$f(-x,y) = -f(x,y), \qquad f(x,-y) = -f(x,y),$$

f is odd in both variables. Hence,

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{nm}\sin(nx)\sin(my).$$

The coefficients are

$$d_{nm} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} xy\sin(nx)\sin(my)\,dx\,dy = \frac{4(-1)^{m+n}}{mn}.$$

Therefore,

$$xy = 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}}{mn}\sin(nx)\sin(my).$$

9. Exercise

Find the double Fourier series of $f(x,y) = -xy$ on $[-\pi,\pi]\times[-\pi,\pi]$.