# Lecture 02 – Part B

Half-Range Fourier Expansion

## 1 Review: Fourier Series on $[-\pi, \pi]$

Let $f$ be a $2\pi$-periodic function integrable on $[-\pi, \pi]$. Its Fourier series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

## 2 Even and Odd Functions

### 2.1 Even Functions

A function $g$ is **even** if

$$g(x) = g(-x).$$

If $g$ is integrable, then

$$\int_{-\pi}^{\pi} g(x)\,dx = 2\int_{0}^{\pi} g(x)\,dx.$$

### 2.2 Odd Functions

A function $g$ is **odd** if

$$g(x) = -g(-x).$$

Then

$$\int_{-\pi}^{\pi} g(x)\,dx = 0.$$

## 3 Fourier Series of Even and Odd Functions

### 3.1 Even Functions

If $f$ is even, then $b_n = 0$ for all $n$ and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

with

$$a_0 = \frac{2}{\pi} \int_{0}^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x)\cos(nx)\,dx.$$

### 3.2 Odd Functions

If $f$ is odd, then $a_0 = a_n = 0$ and

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx),$$

with

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx.$$

## 4 Linearity of Fourier Coefficients

Let $f_1$ and $f_2$ be $2\pi$-periodic functions and $c \in \mathbb{R}$.

- $\mathcal{F}(f_1 + f_2) = \mathcal{F}(f_1) + \mathcal{F}(f_2)$
- $\mathcal{F}(cf) = c\,\mathcal{F}(f)$

This follows from the linearity of integration. Hence, Fourier coefficients form a vector space.

## 5 Example: Sawtooth Wave

Define

$$f(x) = x + \pi, \quad x \in (-\pi, \pi),$$

extended periodically.

Decompose

$$f(x) = x + \pi.$$

The function $x$ is odd and has Fourier series

$$x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx).$$

The constant function $\pi$ has Fourier series $\pi$. Hence,

$$f(x) \sim \pi + 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \cdots$$

## 6 Motivation for Half-Range Expansions

Often a function is defined only on an interval $[0, L]$. Standard Fourier series apply on $[-L, L]$.

**Idea:** Extend $f$ from $[0, L]$ to $[-L, L]$ using symmetry.

## 7 Even Extension (Cosine Series)

Define

$$g(x) = \begin{cases} f(x), & x \in [0, L], \\ f(-x), & x \in [-L, 0]. \end{cases}$$

Then $g$ is even.

The half-range cosine series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

# 8 Odd Extension (Sine Series)

Define

$$g(x) = \begin{cases} f(x), & x \in [0, L], \\ -f(-x), & x \in [-L, 0]. \end{cases}$$

Then $g$ is odd.

The half-range sine series is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

# 9 Example: Triangular Function

Define $f : [0, \pi] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{2}{\pi}x, & 0 \le x \le \frac{\pi}{2}, \\ \frac{2}{\pi}(\pi - x), & \frac{\pi}{2} \le x \le \pi. \end{cases}$$

## Even Extension

The half-range cosine series is

$$f(x) \sim \frac{1}{2} - \frac{16}{\pi^2}\left(\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \cdots\right).$$

## Odd Extension

The half-range sine series is

$$f(x) \sim \frac{8}{\pi^2}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots\right).$$

## 10 Key Takeaways

- Half-range expansions handle functions on $[0, L]$
- Even extension $\Rightarrow$ cosine series
- Odd extension $\Rightarrow$ sine series
- Different series can represent the same function