Lecture 02 – Part A

Fourier Series of a Function with Arbitrary Period

1 Motivation

So far, we have studied the Fourier series of 2π–periodic functions. If f is a 2π–periodic function defined on $(-\pi, \pi)$, then it can be written as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

where the Fourier coefficients are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

However, not every periodic function has period 2π. In this lecture, we extend Fourier series to functions with an *arbitrary period*.

2 Reduction to the 2π–Periodic Case

Let $f(x)$ be a periodic function with period

$$p = 2L, \qquad L > 0.$$

Thus,

$$f(x + 2L) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

2.1 Change of Variables

Define a new variable

$$v = \frac{\pi}{L}x,$$

and introduce a new function

$$g(v) = f(x) = f\left(\frac{L}{\pi}v\right).$$

2.2 Periodicity of g

Since f is $2L$–periodic,

$$g(v+2\pi) = f\left(\frac{L}{\pi}(v+2\pi)\right) = f\left(\frac{L}{\pi}v + 2L\right) = f\left(\frac{L}{\pi}v\right) = g(v).$$

Hence, g is 2π–periodic.

3 Fourier Series for a $2L$–Periodic Function

The Fourier series of $g(v)$ is

$$g(v) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nv) + b_n \sin(nv)\right),$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos(nv)\,dv, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\sin(nv)\,dv.$$

Using $dv = \frac{\pi}{L}\,dx$, we obtain the Fourier series of $f(x)$.

3.1 Fourier Series Formula (Period $2L$)

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\!\left(\frac{n\pi x}{L}\right) + b_n \sin\!\left(\frac{n\pi x}{L}\right)\right],$$

with coefficients

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\!\left(\frac{n\pi x}{L}\right)dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\!\left(\frac{n\pi x}{L}\right)dx, \qquad n \geq 1.$$

4 Example: Periodic Rectangular Wave

Consider the function

$$f(x) = \begin{cases} -k, & -\pi < x < 0, \\ k, & 0 < x < \pi, \end{cases} \qquad f(x+2\pi) = f(x),$$

where $k > 0$.

The function is undefined at $x = 0, \pm\pi$, but this does not affect the Fourier coefficients.

4.1 Properties

- f is 2π–periodic,
- f is piecewise continuous,
- f is odd.

5 Computation of Fourier Coefficients

5.1 Zeroth Coefficient

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-k)\,dx + \int_{0}^{\pi} k\,dx\right) = 0.$$

5.2 Cosine Coefficients

Since f is odd and $\cos(nx)$ is even,

$$a_n = 0 \qquad \text{for all } n \geq 1.$$

5.3 Sine Coefficients

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2k}{n\pi}\left(1 - \cos(n\pi)\right).$$

Hence,

- $b_n = 0$ for even n,
- $b_n = \dfrac{4k}{n\pi}$ for odd n.

6 Fourier Series of the Rectangular Wave

$$f(x) \sim \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

7 Application: Evaluation of an Infinite Series

At a point of continuity, the Fourier series converges to the function value. At $x = \frac{\pi}{2}$,

$$f\left(\frac{\pi}{2}\right) = k.$$

Thus,

$$k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

Cancelling k, we obtain

$$\boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.}$$

8 Concluding Remarks

- Fourier series extend naturally to functions with arbitrary periods.
- A change of variables reduces the problem to the 2π–periodic case.
- Fourier series can be used to compute nontrivial infinite sums.