Lecture 25: Nonhomogeneous Equations

1 Introduction

In this lecture, we discuss **non-homogeneous problems** for partial differential equations (PDEs), including the heat equation, wave equation, and Laplace equation. The techniques presented can be applied to any linear PDE once the homogeneous problem is understood.

2 Relation Between Homogeneous and Nonhomogeneous Equations

Consider a linear operator L and the following two equations:

1. Homogeneous equation:
$$L[W] = 0$$
2. Nonhomogeneous equation:
$$L[V] = F$$

Here, L is **linear in** u, for example:

- $L[u] = u_{xx} + u_{yy}$ (Laplace equation)
- $L[u] = u_{tt} - c^2 u_{xx}$ (wave equation)
- $L[u] = u_t - u_{xx}$ (heat equation)

Key Idea

If:

- W is the general solution of $L[W] = 0$,
- V is a particular solution of $L[V] = F$,

then the general solution of the nonhomogeneous equation is:

$$U = V + W$$

Proof

$$\begin{aligned} L[U] &= L[V+W] \\ &= L[V] + L[W] \quad \text{(linearity)} \\ &= F + 0 \\ &= F \end{aligned}$$

3 Nonhomogeneous Heat Equation

Problem Setup

Consider the heat equation:

$$u_t = u_{xx} + f(x), \quad 0 < x < L,\ t > 0$$

with boundary conditions:

$$u(0,t) = A, \quad u(L,t) = B$$

and initial condition:

$$u(x,0) = f(x)$$

Step 1: Reduce to Homogeneous Problem

Assume a solution of the form:

$$u(x,t) = V(x,t) + U(x)$$

- $U(x)$ handles the boundary condition and nonhomogeneous term.
- $V(x,t)$ solves the homogeneous equation with zero boundary conditions.

Step 2: Substitute into the PDE

$$\begin{aligned} u_t &= V_t \\ u_{xx} &= V_{xx} + U''(x) \end{aligned}$$

Then:

$$V_t = U''(x) + V_{xx} + f(x) \implies V_t = V_{xx} \quad \text{if} \quad U''(x) + f(x) = 0$$

Step 3: Satisfy Boundary Conditions

$$U(0) = A, \quad U(L) = B$$

Then $V(x,t)$ satisfies:

$$V(0,t) = 0, \quad V(L,t) = 0$$

Step 4: Adjust Initial Condition

$$V(x,0) = f(x) - U(x) \equiv h(x)$$

Step 5: Solve Homogeneous Heat Equation

Solve:

$$V_t = V_{xx}, \quad V(0,t) = V(L,t) = 0, \quad V(x,0) = h(x)$$

Step 6: Solve for $U(x)$

$$U(x) = A + \frac{B-A}{L}x + \frac{x}{L}\int_0^L \int_0^\eta f(\xi)d\xi d\eta - \int_0^x \int_0^\eta f(\xi)\, d\xi\, d\eta$$

4 Nonhomogeneous Wave Equation

Problem Setup

$$u_{tt} - u_{xx} = h(x,t), \quad 0 < x < L,\ t > 0$$

with boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0$$

and initial conditions:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

Step 1: Fourier Series Expansion

Assume:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

Expand source term:

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin\left(\frac{n\pi x}{L}\right), \quad h_n(t) = \frac{2}{L}\int_0^L h(x,t)\sin\left(\frac{n\pi x}{L}\right) dx$$

Step 2: Substitute into PDE

$$\sum_{n=1}^{\infty} \left(u_n''(t) + \lambda_n^2 u_n(t) - h_n(t)\right) \sin\left(\frac{n\pi x}{L}\right) = 0, \quad \lambda_n = \frac{n\pi}{L}$$

Step 3: Solve Each ODE

$$u_n''(t) + \lambda_n^2 u_n(t) = h_n(t)$$

Solution:

$$u_n(t) = a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t) + \frac{1}{\lambda_n}\int_0^t h_n(\tau) \sin\left(\lambda_n (t-\tau)\right) d\tau$$

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \quad b_n = \frac{2}{L\lambda_n}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Step 4: Construct Full Solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

5 Non-zero Boundary Conditions

To handle non-zero boundary conditions, assume:

$$u(x,t) = V(x,t) + U(x,t)$$

where $U(x,t)$ satisfies:

$$U(0,t) = P(t), \quad U(L,t) = Q(t)$$

Then $V(x,t)$ satisfies homogeneous boundary conditions and the PDE reduces to:

$$V_{tt} - c^2 V_{xx} = h(x,t) - (U_{tt} - c^2 U_{xx})$$

6 Summary

- Nonhomogeneous PDEs can be solved using superposition: $u = V + W$
- W solves the homogeneous equation
- V is a particular solution of the nonhomogeneous equation
- Reducing to homogeneous boundary conditions simplifies the problem
- Fourier series and eigenfunction expansions allow systematic solutions for arbitrary source terms

Practice Problems

1. Solve $u_t = u_{xx} + x$ on $0 < x < 1$, $u(0,t) = 0, u(1,t) = 1$
2. Solve $u_{tt} - u_{xx} = \sin(x)$ on $0 < x < \pi$, $u(0,t) = u(\pi,t) = 0$
3. Solve $\Delta u = xy$ on a rectangle with Dirichlet boundary conditions