Lecture 24: Heat Equation – Separation of Variables and Uniqueness

1 Introduction

In this lecture, we continue our study of the one-dimensional heat equation. We will cover:

- Separation of variables method.
- Solutions for homogeneous and non-homogeneous boundary conditions.
- Uniqueness theorem for the heat equation using an energy functional.

2 Heat Equation in a Homogeneous Rod

Consider a homogeneous rod of length L with uniform cross-section and density. Let $u(x,t)$ denote the temperature at position x and time t. Assuming the rod is thin and insulated along its surface, the heat equation is given by:

$$u_t = ku_{xx}, \quad 0 < x < L,\ t > 0, \tag{1}$$

with boundary conditions (insulated ends):

$$u(0,t) = u(L,t) = 0, \quad t \geq 0, \tag{2}$$

and initial condition:

$$u(x,0) = f(x), \quad 0 \leq x \leq L. \tag{3}$$

3 Separation of Variables

We look for solutions of the form:

$$u(x,t) = X(x)T(t), \quad u \not\equiv 0.$$

Substituting into (1), we get:

$$X(x)T'(t) = kX''(x)T(t) \implies \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\alpha^2,$$

where $\alpha^2 > 0$ (negative sign ensures non-trivial solutions satisfying the boundary conditions).

This yields two ordinary differential equations:

$$X''(x) + \alpha^2 X(x) = 0, \tag{4}$$

$$T'(t) + k\alpha^2 T(t) = 0. \tag{5}$$

3.1 Boundary Conditions for $X(x)$

From (2):

$$X(0) = 0, \quad X(L) = 0.$$

The general solution of (4) is:

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x).$$

Applying $X(0) = 0$ gives $A = 0$, hence:

$$X(x) = B\sin(\alpha x).$$

Applying $X(L) = 0$ yields:

$$B\sin(\alpha L) = 0 \implies \alpha L = n\pi, \quad n = 1, 2, 3, \ldots$$

Thus the eigenvalues and eigenfunctions are:

$$\alpha_n = \frac{n\pi}{L}, \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

3.2 Time-dependent Part

Solving (5):

$$T_n(t) = e^{-k\alpha_n^2 t} = e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

3.3 Non-trivial Solutions

The non-trivial solutions of (1) are:

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

By the superposition principle, the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}. \tag{6}$$

3.4 Determining Coefficients

Using the initial condition (3), $u(x,0) = f(x)$:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right).$$

The Fourier sine coefficients are:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

4 Non-homogeneous Boundary Condition

If the temperature at $x = L$ is held constant at u_0, i.e.,

$$u(L,t) = u_0,$$

we use a change of variables:

$$u(x,t) = v(x,t) + \frac{u_0}{L}x.$$

Then $v(x,t)$ satisfies the homogeneous boundary conditions:

$$v(0,t) = v(L,t) = 0,$$

and we solve for $v(x,t)$ using the method described above, with the modified initial condition:

$$v(x,0) = f(x) - \frac{u_0}{L}x.$$

The final solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L \left(f(\tau) - \frac{u_0}{L}\tau \right) \sin\left(\frac{n\pi\tau}{L}\right) d\tau \right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) + \frac{u_0}{L}x.$$

5 Uniqueness Theorem

Statement: Let $u(x,t)$ be twice continuously differentiable (C^2) and satisfy the heat equation (1) with initial condition (3) and boundary conditions (2) on a bounded domain $0 < x < L$. Then the solution is unique.

5.1 Proof (Energy Method)

Assume $u_1(x,t)$ and $u_2(x,t)$ are two solutions. Let

$$v(x,t) = u_1(x,t) - u_2(x,t).$$

Then $v(x,t)$ satisfies:

$$v_t = kv_{xx}, \quad v(0,t) = v(L,t) = 0, \quad v(x,0) = 0.$$

Define the energy functional:

$$J(t) = \frac{1}{2} \int_0^L v^2(x,t)\, dx.$$

Differentiating $J(t)$ with respect to t:

$$J'(t) = \int_0^L v v_t \, dx = k \int_0^L v v_{xx} \, dx.$$

Integrating by parts:

$$J'(t) = k \left[v v_x \right]_0^L - k \int_0^L (v_x)^2 \, dx = -k \int_0^L (v_x)^2 \, dx \leq 0.$$

Thus, $J(t)$ is non-increasing. Since $J(0) = 0$, we have $J(t) \equiv 0$, which implies $v(x,t) \equiv 0$.

Conclusion: $u_1(x,t) \equiv u_2(x,t)$. Therefore, the solution is unique.

6 Summary

- Separation of variables yields solutions in terms of eigenfunctions $\sin(n\pi x/L)$ and exponentially decaying time functions.
- Fourier sine series is used to match the initial condition.
- Non-homogeneous boundary conditions can be handled by a suitable change of variables.
- The uniqueness theorem ensures that the solution in a bounded domain is unique using the energy method.