Lecture 23: Heat Equation: Fundamental Solution and Maximum Principle

1 Introduction

In this lecture, we discuss the **heat equation**, which is a prototype of **parabolic partial differential equations**. The general form of the heat equation is

$$u_t - \Delta u = 0,$$

where $u = u(x,t)$ is a function of space $x \in \Omega \subset \mathbb{R}^n$ and time $t \geq 0$, and Δ denotes the Laplacian with respect to x.

If we replace 0 by a function f, we get the **non-homogeneous heat equation**:

$$u_t - \Delta u = f.$$

1.1 Domain and Variables

- Time variable: $t \geq 0$.
- Spatial variable: $x \in \Omega \subset \mathbb{R}^n$, with Ω open.
- Function $u : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$.
- Laplacian is with respect to x only:

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

1.2 1D Heat Equation

In one dimension, the heat equation simplifies to:

$$u_t - u_{xx} = 0, \quad x \in \Omega \subset \mathbb{R},\, t \geq 0.$$

2 Physical Interpretation

The heat equation is also called the **diffusion equation**. It models how quantities such as heat, chemical concentration, or other diffusive substances evolve over time. Key points:

- Unlike the Laplace equation, which describes a **steady state**, the heat equation describes evolution over time.
- Let $V \subset \Omega$ be a smooth subregion. The rate of change of the total heat in V is given by
$$\frac{d}{dt}\int_V u\,dx = -\int_{\partial V} \mathbf{F}\cdot\mathbf{n}\,ds,$$
where $\mathbf{F}$ is the **flux density**.
- By the divergence theorem,
$$\int_V u_t\,dx = -\int_V \nabla\cdot\mathbf{F}\,dx \implies u_t = -\nabla\cdot\mathbf{F}.$$
- If the flux is proportional to the gradient of u in the opposite direction:
$$\mathbf{F} = -a\nabla u, \quad a > 0,$$
then we obtain
$$u_t = a\Delta u.$$

3 Fundamental Solution

[Fundamental Solution of the Heat Equation] The fundamental solution $\Phi(x,t)$ of the heat equation in $\mathbb{R}^n$ is defined as

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} \exp\left(-\dfrac{|x|^2}{4t}\right), & t > 0, \\ 0, & t \le 0. \end{cases}$$

3.1 Solution via Fundamental Solution

For the Cauchy problem
$$\begin{cases} u_t - \Delta u = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^n, \end{cases}$$
the solution can be written as a convolution with the fundamental solution:

$$u(x,t) = (\Phi(\cdot,t) * g)(x) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dy,$$

provided g is bounded and continuous.

3.2 Infinite Speed of Propagation

If $g \geq 0$ and $g \not\equiv 0$, then $u(x,t) > 0$ for all $x \in \mathbb{R}^n$ and $t > 0$. This is called **infinite speed of propagation**, in contrast to the wave equation, which has finite propagation speed.

4 Maximum Principle

4.1 Parabolic Cylinder and Boundary

Define the **parabolic cylinder**:

$$\Omega_T = \Omega \times (0, T),$$

and the **parabolic boundary**:

$$\Gamma_T = \bar{\Omega}_T \setminus \Omega_T = (\partial\Omega \times [0, T]) \cup (\Omega \times \{0\}).$$

4.2 Strong Maximum Principle

Let $u \in C^{2,1}(\Omega_T) \cap C(\bar{\Omega}_T)$ satisfy the heat equation

$$u_t - \Delta u = 0 \quad \text{in } \Omega_T.$$

Then:

1. The maximum of u in $\bar{\Omega}_T$ is attained on the parabolic boundary Γ_T, unless u is constant.
2. Similarly, the minimum of u is attained on Γ_T, unless u is constant.

The principle is intuitive: heat flows from high to low temperature, so the extrema occur on the boundary unless the solution is uniform.

5 Uniqueness in Bounded Domains

If $u_t - \Delta u = f$ in a bounded domain Ω_T with boundary condition $u = g$ on Γ_T, then there exists a unique solution

$$u \in C^{2,1}(\Omega_T) \cap C(\bar{\Omega}_T).$$

5.1 Sketch of Proof

- Assume there exist two solutions u and $\tilde{u}$.
- Let $w = u - \tilde{u}$. Then w satisfies

$$w_t - \Delta w = 0, \quad w = 0 \text{ on } \Gamma_T.$$

- By the strong maximum principle, the maximum and minimum of w occur on the boundary. Hence $w \equiv 0$ and $u = \tilde{u}$.

5.2 Note on Unbounded Domains

In unbounded domains, $\mathbb{R}^n \times (0, \infty)$, the strong maximum principle and uniqueness do not hold without additional growth conditions.

6 Summary

- Heat equation models diffusive phenomena and is parabolic.
- Fundamental solution $\Phi(x, t)$ provides explicit solutions for the Cauchy problem.
- The heat equation exhibits infinite speed of propagation.
- The strong maximum principle ensures that maxima and minima occur on the boundary for bounded domains.
- Uniqueness of solutions holds in bounded domains but not automatically in unbounded domains.