Lecture 01 : Part C
Convergence of Fourier Series

Orthogonality of the Trigonometric System

In this part of the lecture, we discuss the orthogonality of the trigonometric system and explain how this property allows us to compute the coefficients in a Fourier series.

Suppose f is a 2π-periodic function. Our aim is to represent $f(x)$ in terms of an infinite series involving sine and cosine functions, namely

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big). \tag{1}$$

The natural question is: why do we choose sine and cosine functions? Apart from their simplicity, the crucial reason is their *orthogonality*, which enables us to explicitly compute the coefficients a_n and b_n.

The Trigonometric System

The trigonometric system consists of the functions

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots\},$$

considered on the interval $[-\pi, \pi]$. Since all these functions are 2π-periodic, it is sufficient to work on this interval.

Orthogonality Relations

The following orthogonality relations hold on $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0, \quad n \neq m, \tag{2}$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0, \quad n \neq m, \tag{3}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0, \quad \text{for all } n, m. \tag{4}$$

Moreover, for $n \geq 1$,

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \pi, \tag{5}$$

$$\int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi. \tag{6}$$

These identities can be verified using standard trigonometric identities and are left as an exercise.

Determination of Fourier Coefficients

Assume that f is integrable on $[-\pi, \pi]$ and that termwise integration is allowed. We now show how to compute the coefficients in (1).

The Constant Coefficient

Integrating (1) over $[-\pi, \pi]$, we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\,dx.$$

Since

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = 0, \qquad \int_{-\pi}^{\pi} \sin(nx)\,dx = 0,$$

we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0.$$

Hence,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx. \tag{7}$$

The Cosine Coefficients

Fix $m \geq 1$ and multiply (1) by $\cos(mx)$. Integrating over $[-\pi, \pi]$ gives

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = a_m \int_{-\pi}^{\pi} \cos^2(mx)\,dx,$$

where all other terms vanish due to orthogonality. Using (5), we obtain

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(mx)\,dx, \qquad m \geq 1. \tag{8}$$

The Sine Coefficients

Similarly, multiplying (1) by $\sin(mx)$ and integrating over $[-\pi, \pi]$, we obtain

$$\int_{-\pi}^{\pi} f(x) \sin(mx) \, dx = b_m \int_{-\pi}^{\pi} \sin^2(mx) \, dx.$$

Using (6), we conclude

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx, \qquad m \geq 1. \tag{9}$$

The constants a_0, a_n, and b_n are called the *Fourier coefficients* of f.

Fourier Series Representation

Formally, the Fourier series of a 2π-periodic function f is given by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx) \big),$$

where the coefficients are defined by (7), (8), and (9).

The symbol "$\sim$" emphasizes that convergence is not automatic and must be justified.

Convergence Theorem

We now state the main convergence result for Fourier series.

Theorem (Convergence of Fourier Series). Let f be a 2π-periodic function that is piecewise continuous on $[-\pi, \pi]$ and whose left-hand and right-hand derivatives exist at every point of $[-\pi, \pi]$. Then the Fourier series of f converges at every point x.

- If f is continuous at x, then the Fourier series converges to $f(x)$.
- If x_0 is a point of discontinuity of f, then the Fourier series converges to

  $$\frac{1}{2} \left(f(x_0^-) + f(x_0^+) \right),$$

  the average of the left-hand and right-hand limits of f at x_0.

In particular, if f is continuously differentiable and 2π-periodic, then its Fourier series converges to $f(x)$ for all x.

Conclusion

The orthogonality of the trigonometric system allows us to compute Fourier coefficients explicitly and plays a fundamental role in Fourier analysis. Under suitable regularity assumptions, a 2π-periodic function can be recovered from its Fourier series, either pointwise or in terms of averaged limits at discontinuities.