Lecture 22: Nonhomogeneous Wave Equation

1 Introduction

In this lecture, we study the **nonhomogeneous wave equation** in one spatial dimension. Unlike the homogeneous wave equation, a nonhomogeneous wave equation contains a forcing term $f(x, t)$ or $h(x, t)$.

Homogeneous wave equation (review)

$$u_{tt} - c^2 u_{xx} = 0$$

where $u(x, t)$ is the unknown function, and c is the wave speed.

Nonhomogeneous wave equation

$$u_{tt} - c^2 u_{xx} = h^*(x, t)$$

with initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

The spatial domain is $x \in \mathbb{R}$ and time $t \geq 0$.

2 Coordinate Transformation

To simplify the equation, we remove c from the operator by introducing

$$y = ct$$

Then, the wave equation becomes:

$$u_{xx} - u_{yy} = h(x, y)$$

with transformed forcing term and initial conditions:

$$h(x, y) = -\frac{h^*(x, t)}{c^2}, \quad u(x, 0) = f(x), \quad u_y(x, 0) = \frac{g(x)}{c}$$

3   Characteristic Curves

For the 1D wave equation in canonical form:
$$u_{xx} - u_{yy} = h(x, y)$$

- **Characteristic curves**:
$$x \pm y = \text{constant}$$

- For a point $P_0 = (x_0, y_0)$, the characteristics intersect the x-axis at:
$$P_1 = (x_0 - y_0, 0), \quad P_2 = (x_0 + y_0, 0)$$

This forms a triangular domain of dependence in the xy-plane.

4   Solution Using Double Integrals

Start with the wave equation:
$$u_{xx} - u_{yy} = h(x, y)$$

- Integrate over the triangular region R:
$$\iint_R (u_{xx} - u_{yy}) \, dxdy = \iint_R h(x, y) \, dxdy$$

- Apply Green's theorem:
$$\iint_R (u_{xx} - u_{yy}) \, dxdy = \oint_{\partial R} (u_x \, dy + u_y \, dx)$$

- After parameterization and integration along boundaries:
$$u(x_0, y_0) = \frac{1}{2}\left[u(x_0 + y_0, 0) + u(x_0 - y_0, 0)\right] + \frac{1}{2}\int_{x_0 - y_0}^{x_0 + y_0} u_y(x, 0) \, dx - \frac{1}{2}\iint_R h(x, y) \, dxdy$$

Returning to original variable t, the D'Alembert-type formula for the nonhomogeneous wave equation is:

$$\boxed{u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s) \, ds + \frac{1}{2c}\iint_R h^*(\xi, \tau) \, d\xi d\tau}$$

5   Example

Solve:
$$u_{xx} - u_{yy} = 1, \quad u(x, 0) = \sin x, \quad u_y(x, 0) = x$$

- Characteristics: $x + y = \text{constant}, \quad x - y = \text{constant}$
- Using the formula:
$$u(x_0, y_0) = \frac{1}{2}[\sin(x_0 + y_0) + \sin(x_0 - y_0)] + \frac{1}{2}\int_{x_0 - y_0}^{x_0 + y_0} s \, ds - \frac{1}{2}\iint_R 1 \, dxdy$$

- After integration:

$$\boxed{u(x_0, y_0) = \frac{1}{2}[\sin(x_0 + y_0) + \sin(x_0 - y_0)] + x_0 y_0 - \frac{1}{2}y_0^2}$$

6 Physical Interpretation

- $u(x,t)$ represents the **displacement** of a string or membrane at position x and time t.
- Initial displacement: $u(x,0) = f(x)$
- Initial velocity: $u_t(x,0) = g(x)$
- Nonhomogeneous term $h^*(x,t)$ represents external forcing.
    Relation to Newton's law:

$$\text{Mass density} \cdot \text{acceleration} = \text{net force} \implies u_{tt} = \text{divergence of force}$$

   - Linearization for small displacements leads to the wave equation.

7 Uniqueness Theorem

For the nonhomogeneous wave equation in a bounded domain $u = (a,b)$ with initial and boundary data:

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad u(x,0) = g(x), \quad u_t(x,0) = h(x)$$

Theorem: There exists at most one solution $u \in C^2(\overline{u} \times [0,T])$.

Proof (Sketch)

1. Suppose u and $\tilde{u}$ are two solutions. Define $w = u - \tilde{u}$.

2. w satisfies the homogeneous problem:

$$w_{tt} - c^2 w_{xx} = 0, \quad w(x,0) = w_t(x,0) = 0, \quad w = 0 \text{ on boundaries}$$

3. Define energy:
$$E(t) = \frac{1}{2} \int_a^b \left(w_t^2 + c^2 w_x^2 \right) dx$$

4. Differentiate $E(t)$ and apply integration by parts: $E'(t) = 0 \implies E(t) = E(0) = 0$

5. Hence $w \equiv 0$, proving uniqueness.

8 Summary

- Nonhomogeneous wave equations include a forcing term $f(x,t)$. - D'Alembert formula generalizes to include the forcing term via a double integral over the triangular domain defined by characteristics. - Energy method proves uniqueness in bounded domains. - Physical interpretation: displacement of a string or elastic medium; Newton's law connects acceleration with net force. - Approach generalizes to 2D and 3D wave equations, heat, and Laplace equations.