Lecture 21: Wave Equation and D'Alembert Formula

1 Introduction

In this lecture, we study the one-dimensional wave equation and derive the D'Alembert formula, which provides a solution to the Cauchy problem for the wave equation on the real line.

2 Wave Equation on the Real Line

Consider the one-dimensional wave equation:

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

where $c > 0$ is the speed of the wave.

2.1 Initial Conditions

For the Cauchy problem, we are given initial conditions:

$$u(x, 0) = f(x), \tag{2}$$
$$u_t(x, 0) = g(x), \tag{3}$$

where f and g are smooth functions, typically assumed $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$.

3 Method of Characteristics

The wave equation is hyperbolic. Its characteristic curves are given by:

$$x \pm ct = \text{constant}. \tag{4}$$

Introduce the characteristic variables:

$$\xi = x + ct, \tag{5}$$
$$\eta = x - ct. \tag{6}$$

Using the chain rule, the wave equation transforms as:

$$u_{\xi\eta} = 0. \tag{7}$$

3.1 General Solution in Characteristic Form

The general solution of $u_{\xi\eta} = 0$ is

$$u(\xi, \eta) = \phi(\xi) + \psi(\eta), \tag{8}$$

where ϕ and ψ are arbitrary smooth functions.

Returning to (x, t) variables, we have:

$$u(x, t) = \phi(x + ct) + \psi(x - ct). \tag{9}$$

4 Applying Initial Conditions

Using $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$:

$$f(x) = \phi(x) + \psi(x), \tag{10}$$
$$g(x) = c\phi'(x) - c\psi'(x). \tag{11}$$

Solving for ϕ and ψ:

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\tau)\, d\tau + \frac{k}{2}, \tag{12}$$
$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\tau)\, d\tau - \frac{k}{2}, \tag{13}$$

where x_0 and k are arbitrary constants.

4.1 D'Alembert's Formula

Combining ϕ and ψ, the solution is given by the D'Alembert formula:

$$\boxed{u(x, t) = \frac{1}{2}\big[f(x + ct) + f(x - ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau}. \tag{14}$$

5 Remarks on the Solution

- **Domain of Dependence:** The value of $u(x_0, t_0)$ depends only on the initial data f at $x_0 \pm ct_0$ and g on the interval $[x_0 - ct_0, x_0 + ct_0]$. Values outside this interval do not influence the solution.

- **Range of Influence:** The initial data at a point $(x_0, 0)$ influences the solution only in the sector bounded by the characteristic lines $x - x_0 = \pm ct$.

- **Continuous Dependence on Initial Data:** If f and g are perturbed slightly, the solution u changes slightly. More precisely, for any $\epsilon > 0$ and time interval $[0, t_0]$, there exists $\delta > 0$ such that:

  $$\|f - f^*\|_\infty < \delta, \quad \|g - g^*\|_\infty < \delta \implies \|u - u^*\|_\infty < \epsilon,$$

  where u^* is the solution corresponding to perturbed initial data f^*, g^*.

6 Example

Consider the problem:

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, t > 0, \tag{15}$$

$$u(x, 0) = \sin(x), \tag{16}$$

$$u_t(x, 0) = \cos(x). \tag{17}$$

Applying D'Alembert's formula:

$$\begin{aligned} u(x,t) &= \frac{1}{2}\big[\sin(x+ct) + \sin(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos(\tau)\, d\tau \\ &= \sin(x)\cos(ct) + \frac{1}{c}\sin(ct)\cos(x) \\ &= \sin(x)\cos(ct) + \cos(x)\frac{\sin(ct)}{c}. \end{aligned}$$

7 Domain of Dependence and Range of Influence

For a point (x_0, t_0):

- **Domain of Dependence:** $[x_0 - ct_0, x_0 + ct_0]$ for g and $\{x_0 - ct_0, x_0 + ct_0\}$ for f.
- **Range of Influence:** The solution at (x_0, t_0) influences points in the sector bounded by characteristics $x - x_0 = \pm c(t - t_0)$.

8 Conclusion

D'Alembert's formula provides an explicit solution to the Cauchy problem for the one-dimensional wave equation. The solution depends only on initial data within the domain of dependence, and any perturbation in the initial data leads to a controlled change in the solution, ensuring well-posedness.