Lecture 20: Wave Equation
Separation of Variables

1 Introduction to the Wave Equation

In this lecture, we study the **wave equation**, which is a second–order linear partial differential equation. The simplest form of the one–dimensional wave equation is

$$u_{tt} - u_{xx} = 0. \tag{1}$$

This equation is known as the **one–dimensional homogeneous wave equation**.

The function $u(x, t)$ depends on two variables:

- x: the spatial variable,
- $t \geq 0$: the time variable.

More generally, the wave equation may be written as

$$u_{tt} - c^2 u_{xx} = 0, \tag{2}$$

where $c > 0$ is a constant representing the wave speed.

1.1 Higher–Dimensional Wave Equations

In higher dimensions, the wave equation takes the form

$$u_{tt} - \Delta u = 0, \tag{3}$$

where Δ denotes the Laplacian operator. For example:

$$\text{2D: } u_{tt} - (u_{xx} + u_{yy}) = 0,$$
$$\text{3D: } u_{tt} - (u_{xx} + u_{yy} + u_{zz}) = 0.$$

2 Homogeneous vs Inhomogeneous Wave Equation

Let

$$L(u) = u_{tt} - u_{xx}.$$

- If $L(u) = 0$, the equation is called the **homogeneous wave equation**.
- If $L(u) = g(x, t)$ for some function g, the equation is called the **inhomogeneous wave equation**.

In this lecture, we focus on the homogeneous case.

3 Principle of Superposition

Theorem (Superposition Principle). If u_1 and u_2 are solutions of the homogeneous wave equation in a region R, then for any constants $C_1, C_2 \in \mathbb{R}$,

$$u = C_1 u_1 + C_2 u_2 \tag{4}$$

is also a solution in R.

3.1 Proof (Sketch)

Since the wave equation is linear,

$$L(C_1 u_1 + C_2 u_2) = C_1 L(u_1) + C_2 L(u_2) = 0.$$

Hence, linear combinations of solutions are also solutions.

4 Physical Interpretation

The one–dimensional wave equation models the motion of a vibrating elastic string.

- $u(x,t)$ represents the vertical displacement of the string.
- The string is stretched along $0 \leq x \leq L$.
- The ends of the string are fixed.

5 Boundary and Initial Conditions

We consider the problem

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L,\ t > 0, \tag{5}$$

subject to:

5.1 Boundary Conditions (Fixed Ends)

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0. \tag{6}$$

5.2 Initial Conditions

$$u(x,0) = f(x), \quad \text{(initial displacement)} \tag{7}$$
$$u_t(x,0) = g(x), \quad \text{(initial velocity)} \tag{8}$$

6 Separation of Variables

Assume a solution of the form

$$u(x,t) = F(x)G(t). \tag{9}$$

Substituting into the wave equation gives

$$\frac{G''(t)}{c^2G(t)} = \frac{F''(x)}{F(x)} = -\lambda,$$

where λ is a separation constant.

This leads to two ordinary differential equations:

$$F'' + \lambda F = 0, \tag{10}$$

$$G'' + c^2\lambda G = 0. \tag{11}$$

7 Spatial Eigenvalue Problem

Using the boundary conditions,

$$F(0) = 0, \quad F(L) = 0,$$

we obtain a Sturm–Liouville problem whose nontrivial solutions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \tag{12}$$

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots \tag{13}$$

8 Temporal Equation

For each n, the time–dependent equation becomes

$$G_n'' + c^2\left(\frac{n\pi}{L}\right)^2 G_n = 0. \tag{14}$$

Its general solution is

$$G_n(t) = A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right). \tag{15}$$

9 General Solution

Using superposition, the solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)\right]\sin\left(\frac{n\pi x}{L}\right). \tag{16}$$

10 Determination of Coefficients

10.1 From Initial Displacement

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right),$$

so

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \tag{17}$$

10.2 From Initial Velocity

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right),$$

so

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \tag{18}$$

11 Special Case: Zero Initial Velocity

If $g(x) \equiv 0$, then $B_n = 0$ and

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi c}{L}t\right)\sin\left(\frac{n\pi x}{L}\right). \tag{19}$$

12 Example

Let

$$f(x) = \begin{cases} \frac{2k}{L}x, & 0 < x < \frac{L}{2}, \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L, \end{cases} \qquad g(x) = 0.$$

Then the solution is

$$u(x,t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi ct}{L}\right) + \cdots\right]. \tag{20}$$

13 Conclusion

In this lecture, we:

- Introduced the wave equation,
- Discussed physical interpretation,
- Applied separation of variables,

- Used Fourier series to satisfy initial conditions.

In the next lecture, we will study further properties of the wave equation and additional examples.